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Wild ramification kinks

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Abstract

Given a branched cover $f: Y \to X$ between smooth projective curves over a non-archimedean mixed-characteristic local field and an open rigid disk $D \subset X$, we study the question under which conditions the inverse image $f^{-1}(D)$ is again an open disk. More generally, if the cover f varies in an analytic family, is this true at least for some member of the family? Our main result gives a criterion for this to happen.

Keywords: p-adic disk, Berkovich different, Swan conductor, Rigid-analytic Galois cover

Mathematics Subject Classification: 14H30, 14H37, 14D99, 14G22, 11G20

1 Background

This paper is about inverse images of non-archimedean disks under finite morphisms—specifically, when are they themselves disks?

Let X be a smooth, projective curve over a mixed-characteristic $(0, p)$ non-archimedean field K. Our main result (Theorem 5.2) applies to flat families $\mathcal{F}: \mathcal{Y} \to X \times \mathcal{A}$ of Galois branched covers of X parameterized by a quasi-compact, quasi-separated, rigid-analytic space $\mathcal{A}$ (e.g., an affinoid). Let D be an open disk in X (say, given a choice of origin and a metric making it a unit disk), and for $r > 0$, let $D[r] \subseteq D$ be the closed disk centered at the origin of radius p^{-r} (i.e., the set of all points with valuation at least r). The theorem says that, under mild assumptions about branch loci and connectedness (see the beginning of Sect. 5), if there exists a sequence $r_1, r_2, \ldots$ decreasing to 0 and points $a_1, a_2, \ldots$ in $\mathcal{A}$ such that $(\mathcal{F}|_{a_i})^{-1}(D[r_i])$ is a closed disk for all i, then there exists $a \in \mathcal{A}$ such that $(\mathcal{F}|_a)^{-1}(D)$ is an open disk. In fact, the main result is slightly more general, allowing X to vary over $\mathcal{A}$ in a somewhat prescribed manner (see Assumption 4.10) and allowing $\mathcal{F}$ to be a *tower* of Galois covers.

While the above problem is of intrinsic interest as a statement about rigid geometry, we are mainly motivated by the *local lifting problem*, which asks whether a given action of a finite group G on the germ of a smooth curve in characteristic p lifts to characteristic 0. Our recent paper [19] introduces a new "iterative" technique for solving this problem and solves it when G is cyclic, proving the *Oort conjecture*. What we prove here generalizes a key technical step from [19] needed for the iterative technique to work. The generality we work in applies, for instance, to [18], which builds on [19] to examine the local lifting problem for metacyclic groups. Indeed, we expect that Theorem 5.2 is sufficiently general to be useful in any solution to the local lifting problem that proceeds via the iterative technique from [19].

The key idea in this paper is to rephrase the question of whether the inverse image of a disk is a disk in two different ways: one in terms of Cohen–Temkin–Trushin's *Berkovich different* [9] and one in terms of Kato's *depth Swan conductor* [15]. Indeed, Theorem 5.2 can be interpreted in terms of either the Berkovich different or Swan conductor, as is done in Corollary 5.3 (the Swan conductor version is the form of the theorem applied in [18,19]). The maximal r for which the inverse image of $D[r]$ is not a disk corresponds to a kink in a piecewise linear function built from the Berkovich different/Swan conductor, hence the title of the paper. The location of this kink (that is, r) can be detected from valuations of certain analytic functions in the coefficients of the polynomials defining the cover. For a family of covers parameterized by a quasi-compact, quasi-separated $\mathcal{A}$, the maximum principle for absolute values guarantees that r achieves its infimum on $\mathcal{A}$, which shows that there is some $a \in \mathcal{A}$ where a kink does not appear for any $r > 0$. This suffices to prove the main result.

In Sect. 2, we introduce the Berkovich different and depth and differential Swan conductors and relate them to the problem of whether the inverse image of a disk is a disk. The main result is Corollary 2.20, which depends on a genus formula of Cohen–Temkin–Trushin ([9], which we apply as Proposition 2.7). This can also be seen as a consequence of a vanishing cycles result of Kato [16]. In Sect. 3, we compute the Swan conductors of a $\mathbb{Z}/p$-cover explicitly in terms of Kummer representatives, generalizing work in [19]. In Sect. 4, we examine relative cyclic covers parameterized by rigid-analytic spaces. Corollary 4.21 proves our main result in the case of a $\mathbb{Z}/p$-cover. Lastly, we put everything together in Sect. 5 to prove the main result for general towers of Galois covers.

1.1 Notation/conventions

If G is a finite group, then a *character* on G means the character of a finite-dimensional $\mathbb{C}$-representation of G. A *faithful* (resp. *irreducible*) character is one that corresponds to a faithful (resp. irreducible) representation.

Throughout, R is a complete discrete valuation ring with fraction field K of characteristic 0 and algebraically closed residue field k of characteristic p. The field C is the completion of an algebraic closure of K. We will often replace K and R with finite extensions inside C without changing the notation.

If X is a projective curve over K, then we write X^{an} (resp. X^{Berk}) for the rigid-analytic (resp. Berkovich) space corresponding to X (resp. to $X \times_K C$). Similarly, if $f\colon Y \to X$ is a morphism of projective curves over K, we write f^{an} and f^{Berk} for the corresponding rigid-analytic and Berkovich morphisms.

A *closed* (*rigid-analytic*) disk is a rigid-analytic space isomorphic to $\mathrm{Sp}\, K\{T\}$, where

$$K\{T\} := \left\{ \sum_{i=0}^{\infty} a_i T^i \mid a_i \in K,\ a_i \to 0 \right\}.$$

An *open* (*rigid-analytic*) *disk* is a rigid-analytic space isomorphic to the admissible open inside $\mathrm{Sp}\, K\{T\}$ given by $|T| < 1$.

For a rigid-analytic space, the property of being quasi-compact and quasi-separated will be abbreviated to qcqs.

Remark 1.1 It seems plausible that our main result should also hold in equal characteristic and should also hold without requiring K to be discretely valued. The first generalization will require significantly different techniques, as our proof is heavily based on Kummer theory.

2 Ramification of Galois extensions in mixed characteristic

Throughout Sect. 2, we fix a branched cover (i.e., a finite, surjective K-morphism) $f: Y \to X$ of smooth, projective, geometrically connected K-curves.

We mention that type 2 points on X^{Berk} correspond to irreducible components of semistable models of X over some finite extension of K, and vice versa (this follows, for instance, from [7, Theorem 4.11]). We will make frequent use of this correspondence.

2.1 The different of Cohen, Temkin, and Trushin

For each point y in Y^{Berk}, Cohen–Temkin–Trushin define the *different* δ_y of f^{Berk} at y ([9, §2.4.1 and Definition 4.1.2]—we only need the definition at type 2 and 3 points). Namely, if T and S are the valuation rings of the completed residue fields of y and $f^{\mathrm{Berk}}(y)$, respectively, then $\delta_y = |\mathrm{Ann}(\Omega_{T/S})|$, where $|I| = \sup_{a\in I} |a|_y$ for an ideal $I \subseteq T$. Note that this is a special case of a more general definition due to Gabber and Romero, see [9, Remark 2.4.2]. We use the notation $\delta^{\mathrm{Berk}}_{Y/X,y} := \log_{|p|}(\delta_y)$ (viewing the different as a valuation, rather than as an absolute value). We will write δ^{Berk}_y instead of $\delta^{\mathrm{Berk}}_{Y/X,y}$ when $Y \to X$ is understood. Note that $\delta^{\mathrm{Berk}}_y = 0$ when T/S is unramified.

The different behaves nicely in towers:

Proposition 2.1 ([9, Corollary 2.4.5]) *Suppose $Y = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = X$ is a tower of branched covers of smooth, geometrically connected projective curves over K. For each $0 \le i \le n$, pick $y_i \in Y_i^{\mathrm{Berk}}$ of type 2 or 3 such that $y_i \mapsto y_{i-1}$. Then $\delta^{\mathrm{Berk}}_{Y/X,y_n} = \sum_{i=1}^n \delta^{\mathrm{Berk}}_{Y_i/Y_{i-1},y_i}$.*

Note that if $f: Y \to X$ is given as a finite composition of *Galois* covers, then δ_y depends only on $f^{\mathrm{Berk}}(y)$. In this case, if $x \in X^{\mathrm{Berk}}$, we define δ^{Berk}_x to be δ^{Berk}_y for any $y \in (f^{\mathrm{Berk}})^{-1}(x)$.

2.2 Kato's Swan conductors

In this section, suppose that $f: Y \to X$ is G-*Galois*, for G a finite p-group, and that χ is a character of G.

Let X_R be a semistable model of X defined over R with special fiber $\bar{X}$ (such a model exists after a finite extension of K, see e.g., [10]). After a further finite extension of K, we may assume that the normalization Y_R of X_R in $K(Y)$ has reduced special fiber $\bar{Y}$ [11]. Let $\bar{V}$ be an irreducible component of $\bar{X}$ with generic point $\eta_{\bar{V}}$, and let $\bar{W}$ be an irreducible component of $\bar{Y}$ with generic point $\eta_{\bar{W}}$ lying above $\bar{V}$.

Assume that the extension $k(\bar{W})/k(\bar{V})$ is *purely inseparable* (and non-trivial). Then the extension $\hat{\mathcal{O}}_{Y_R,\eta_{\bar{W}}}/\hat{\mathcal{O}}_{X_R,\eta_{\bar{V}}}$ is a “Case II” extension in the sense of [15]. Thus we may define the associated *depth Swan conductor* $\delta_{\bar{V}}(\chi) \in \mathbb{Q}_{>0}$ ([8, Definition 1.5.2], but we normalize the valuation so that p has valuation 1). Furthermore, if χ has degree 1, then [8, Theorem 1.5.2] defines the *differential Swan conductor* $\omega_{\bar{V}}(\chi)$, which is a meromorphic differential form on $\bar{V}$, well-defined once a uniformizer of $\hat{\mathcal{O}}_{X_R,\eta_{\bar{V}}}$ is chosen (that this lives in $\Omega^1_{k(\bar{V})}$ instead of some higher tensor power is due to [15, Theorem 3.6]). We will always implicitly make this choice of uniformizer, and it will never be relevant. It follows from their definitions that these Swan conductors are invariant under further extensions of K. If we need to specify the cover, we will write $\delta_{Y/X,\bar{V}}(\chi)$ and $\omega_{Y/X,\bar{V}}(\chi)$.

If, on the other hand, $k(\bar{W})/k(\bar{V})$ is *separable*, we set $\delta_{\bar{V}}(\chi) = 0$ and we do not define the differential Swan conductor.

2.3 Comparison between depth Swan conductor and different

The depth Swan conductor and the different are closely related. For our main result, we only need to understand this relation for $\mathbb{Z}/p$-covers, but we go a bit deeper here to be able to phrase our main result in terms of Swan conductors (Corollary 5.3), as is the context in [18,19].

As in Sect. 2.2, we assume that $f: Y \to X$ is a G-Galois cover, with G a *p-group*, and that χ is a character of G. Let $x \in X^{\mathrm{Berk}}$ be a type 2 point such that $(f^{\mathrm{Berk}})^{-1}(x)$ consists of a single point $y \in Y^{\mathrm{Berk}}$. Then y is a type 2 point. Let δ_x^{Berk} be defined as in Sect. 2.1. After a finite extension of K, there exists a semistable model X_R of X whose special fiber $\bar{X}$ has an irreducible component $\bar{V}$ corresponding to x. After a further finite extension of K, we may assume that the normalization of X_R in $K(Y)$ gives a semistable model Y_R of Y with a unique irreducible component $\bar{W}$ lying above $\bar{V}$. We assume $k(\bar{W})/k(\bar{V})$ is either separable ("the separable case") or purely inseparable ("the purely inseparable case"), and we let $\delta_{\bar{V}}(\chi)$ be the depth Swan conductor as in Sect. 2.2.

Kato defines a different for "Case II" extensions in [15, §2]. The relation with δ^{Berk} is as follows:

Lemma 2.2 *If we are in the purely inseparable case, then δ_x^{Berk} is the same as the valuation δ of the "non-differential" part of Kato's different $\mathcal{D}(\mathrm{Frac}(\hat{\mathcal{O}}_{Y,\eta_{\bar{W}}})/\mathrm{Frac}(\hat{\mathcal{O}}_{X,\eta_{\bar{V}}}))$.*

Proof The extension $\hat{\mathcal{O}}_{Y,\eta_{\bar{W}}}/\hat{\mathcal{O}}_{X,\eta_{\bar{V}}}$ is an extension of complete discrete valuation rings and thus has a different whose valuation is by definition equal to δ. Once K is large enough so that this extension is weakly unramified, then base changing from K to C does not affect δ (for $\mathbb{Z}/p$-extensions, this is a consequence of [14, Proposition 1.6], for example, and then follows in general from the behavior of differents in towers). Since δ can be defined in the same way as δ_x^{Berk} once we have base changed to C, it is equal to δ_x^{Berk}. □

Lemma 2.3 *If $G = \mathbb{Z}/p$ and χ is a faithful character on G of degree 1, then $\delta_{\bar{V}}(\chi) = p\delta_x^{\mathrm{Berk}}/(p-1)$.*

Proof If we are in the purely inseparable case, this follows from [8, Lemma 1.4.5], combined with Lemma 2.2. If we are in the separable case, then $k(\bar{W})/k(\bar{V})$ is separable and is the same as the residue field extension of T/S from Sect. 2.1. Thus $\delta_x^{\mathrm{Berk}} = \delta_{\bar{V}}(\chi) = 0$.

Proposition 2.4 *Suppose $G = \mathbb{Z}/p^n$ and χ is a faithful character on G of degree 1. Let $h: Z \to X$ be the intermediate subcover of $Y \to X$ of degree p^{n-1} and let z be the image of y in Z. Then*

$$\delta_{\bar{V}}(\chi) = \delta_{Z/X,x}^{\mathrm{Berk}} + \frac{p}{p-1}\delta_{Y/Z,y}^{\mathrm{Berk}}.$$

Proof Let H be the unique subgroup of G of order p. Let ψ be a faithful character on H of degree 1. Then $\mathrm{Ind}_H^G(\psi)$ is a sum of p^{n-1} faithful characters of G of degree 1, which all must have the same depth Swan conductor at χ. So $\delta_{\bar{V}}(\mathrm{Ind}_H^G(\psi)) = p^{n-1}\delta_{\bar{V}}(\chi)$. The proposition now follows by [15, Proposition 3.3(2)], combined with Lemmas 2.2 and 2.3.

□

Now, since G is a p-group, it has a composition series $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$ with all successive quotients isomorphic to $\mathbb{Z}/p$. Thus we can break the cover $f\colon Y \to X$ up into a tower of $\mathbb{Z}/p$-covers $Y =: Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 := X$, where $Y_i = Y/G_i$. For $0 \le i \le n$, let y_i be the image of y in Y_i^{Berk} (equivalently, y_i is the unique preimage of x in Y_i^{Berk}). All y_i are type 2 points.

Recall that if G is supersolvable, then any irreducible representation is induced from a degree 1 representation on a cyclic subgroup H of G [22, §8.5, Theorem 16]. In particular, this holds for G a p-group.

Proposition 2.5 *Suppose G is an arbitrary p-group as above and χ is an arbitrary irreducible character on G. Then there exists a composition series of G as above, as well as nonnegative rational numbers $c_1, \ldots, c_n$ such that*

$$\delta_{\bar{V}}(\chi) = \sum_{i=1}^{n} c_i \delta^{\mathrm{Berk}}_{Y_i/Y_{i-1}, y_i}.$$

The composition series and c_i depend only on G and χ (not on f). In particular, if χ is faithful and is induced from a degree 1 character ψ on a subgroup $H \subseteq G$ having index p^m, then if the composition series includes H, we have $c_1 = \cdots = c_{n-1} = p^m$ and $c_n = p^{m+1}/(p-1)$.

Proof Let $M \subseteq G$ be the kernel of the representation corresponding to χ. Then $\delta_{\bar{V}}(\chi)$ does not change when Y is replaced by Y/M and χ is descended to G/M [15, Proposition 3.3(1)]. By choosing a composition series in which M appears as some G_j, setting $c_i = 0$ for $i > j$, and replacing Y by Y/M, we may assume χ is faithful.

Let $Z = Y/H$. By [15, Proposition 3.3(2)] and Lemma 2.2, we have

$$\delta_{Y/X,\bar{V}}(\chi) = p^m \left(\delta_{Y/Z,\bar{W}}(\psi) + \delta^{\mathrm{Berk}}_{Z/X,x} \right)$$

(here $\bar{W}$ is the irreducible component of the stable reduction of Z lying above $\bar{V}$). By taking a composition series that includes H, we obtain the proposition from Propositions 2.4 and 2.1. □

2.4 A local vanishing cycles formula

For Sect. 2.4, we assume that $f\colon Y \to X$ is a G-Galois cover with G a *cyclic* p-group of order p^n, and we let χ be a degree 1 faithful character on G. Let $\mathbb{B}$ be the branch locus of f, and assume that K is large enough so that each branch point has degree 1. Let X'_R be a flat model of X over R with *integral* and *unibranched* special fiber $\bar{X}'$ (this may require another finite extension of K). Let Y'_R be the normalization of X'_R in $K(Y)$, and let $\bar{Y}'$ be the special fiber of Y'_R. After a further extension of K, we may assume that $\bar{Y}'$ is reduced. Let us further assume that $\bar{Y}'$ is irreducible and $k(\bar{Y}')/k(\bar{X}')$ is purely inseparable. Let Z'_R be the quotient of Y'_R by the unique subgroup of order p, and let $\bar{Z}'$ be the special fiber of Z'_R.

Let $q_X\colon \tilde{\bar{X}}' \to \bar{X}'$ denote the normalization of $\bar{X}'$, and likewise for $q_Y\colon \tilde{\bar{Y}}' \to \bar{Y}'$ and $q_Z\colon \tilde{\bar{Z}}' \to \bar{Z}'$. For $\bar{x} \in \bar{X}'$, set

$$\delta_{\bar{x}} := \dim_k \left((q_X)_* \mathcal{O}_{\tilde{\bar{X}}'} / \mathcal{O}_{\bar{X}'} \right)_{\bar{x}}, \tag{2.6}$$

and similarly for $\bar{y} \in \bar{Y}'$ and $\bar{z} \in \bar{Z}'$. For $\bar{x} \in \bar{X}'$, let $U(\bar{x}) \subset X^{\mathrm{an}}$ be the set of all points specializing to $\bar{x}$ (for the model X'_R). Lastly, if $X_R \to X'_R$ is a blowup such that X_R is a semistable model of X and $\bar{V} \subseteq X_R$ is the strict transform of $\bar{X}'$, let $\omega_{\bar{V}}(\chi)$ be the differential Swan conductor from Sect. 2.2. Note that $\bar{V}$ can be canonically identified with $\tilde{\bar{X}}'$, so we can think of $\omega_{\bar{V}}(\chi)$ as a meromorphic differential form on $\tilde{\bar{X}}'$.

Proposition 2.7 *With the notation introduced above, let $\bar{y} \in \bar{Y}'$ and $\bar{z} \in \bar{Z}'$ be the unique points lying above $\bar{x}$. If $n \geq 2$ we have*

$$\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_{\bar{V}}(\chi)) = \frac{2}{p^{n-1}(p-1)}(\delta_{\bar{y}} - \delta_{\bar{z}}) - 2\delta_{\bar{x}} - |\mathbb{B} \cap U(\bar{x})|. \tag{2.8}$$

If $n = 1$ we have

$$\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_{\bar{V}}(\chi)) = \frac{2}{p-1}\delta_{\bar{y}} - \frac{2p}{p-1}\delta_{\bar{x}} - |\mathbb{B} \cap U(\bar{x})|. \tag{2.9}$$

Proof Equation (2.9) follows from [9, Theorem 6.2.7], where, in the notation of [9], $g(V)$, $g(U)$, n, n_y, R_y, $\mathrm{Ram}(f)$, and S_y are, respectively, equal to $\delta_{\bar{y}}$, $\delta_{\bar{x}}$, p, p, $p-1$, $\mathbb{B} \cap U(\bar{x})$, and $(1-p)\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_{\bar{V}}(\chi))$. The only identification that requires explanation is that for S_y. The definition of S_y in [9, §1.3.2] shows that $S_y = p - 1 - m$, where m is the slope of δ^{Berk} on the space Y^{Berk} in the direction corresponding to $q_Y^{-1}(\bar{y})$ at the type 2 point corresponding to $\tilde{\bar{Y}}'$. Using Lemma 2.3 and Proposition 2.13 below (which does not depend on this proposition), we see that $m = (p-1)(\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_{\bar{V}}(\chi)) + 1)$, yielding the desired formula for S_y.

We omit the proof of (2.8), since it will not be used in the sequel. However, see Remarks 2.10 and 2.23.

Remark 2.10 We call the formulas in Proposition 2.7 "local vanishing cycles formulas" because one can also derive them from Kato's vanishing cycles formula (see the more complicated proof of Proposition 2.7 in the earlier version [20] of this article).

Remark 2.11 The formulas in Proposition 2.7 correct the erroneous formula from [19, Proposition 5.12]. However, [19, Proposition 5.12] is used only in [19, Corollaries 5.13 and 5.14], and both of these corollaries also follow from the corrected version above. Thus, the error in [19, Proposition 5.12] does not affect the sequel of that paper.

2.5 Disks inside curves

The above phenomena will be particularly relevant to us when the irreducible components in question correspond to closed disks.

2.5.1 Geometric setup

Suppose $D \subset X^{\mathrm{an}}$ is an open disk. After an extension of K, we can find a semistable model X_R of X whose special fiber $\bar{X}$ contains a *smooth* point $\bar{x}_0$ such that D is the set of points of X^{an} specializing to $\bar{x}_0 \in \bar{X}$. Conversely, if X_R is a semistable model of X with special fiber $\bar{X}$ and $\bar{x}_0 \in \bar{X}$ is smooth, then the set of points specializing to $\bar{x}_0$ is isomorphic to an open disk [4].

To make this isomorphism explicit, we choose some $x_0 \in X(K)$ specializing to $\bar{x}_0$ and an element $T \in \mathcal{O}_{X_R,\bar{x}_0}$ with $T(x_0) = 0$ and whose restriction to the special fiber generates the maximal ideal of $\mathcal{O}_{\bar{X},\bar{x}_0}$ (this is possible because $X_R \to \mathrm{Spec}\, R$ is smooth). Then $\hat{\mathcal{O}}_{X_R,\bar{x}_0} = R[[T]]$, and T induces an isomorphism of rigid-analytic spaces

$$D \cong \left\{ x \in (\mathbb{A}^1_K)^{\mathrm{an}} \mid v(x) > 0 \right\},$$

which sends the point x_0 to the origin. We call T a *parameter* for the open disk D. The choice of T having been made, we identify D with the above subspace of $(\mathbb{A}^1_K)^{\mathrm{an}}$.

For $r \in \mathbb{Q}_{\geq 0}$ we define

$$D[r] := \left\{ x \in D \mid v(x) \geq r \right\} \quad \text{and} \quad D(r) := \left\{ x \in D \mid v(x) > r \right\}.$$

For $r \in \mathbb{Q}_{>0}$ the subset $D(r)$ (resp. $D[r]$) of D is an open disk (resp. is an affinoid subdomain and a closed disk). Let $v_r\colon K(X)^\times \to \mathbb{Q}$ denote the "Gauss valuation" with respect to $D[r]$. This is a discrete valuation on $K(X)$ which extends the valuation v on K and has the property $v_r(T) = r$. It corresponds to the supremum norm on the open subset $D[r] \subset X^{\mathrm{an}}$. Let κ_r denote the residue field of $K(X)$ with respect to the valuation v_r. After replacing K by a finite extension (which depends on r), we may assume that $p^r \in K$. Then $D[r]$ is isomorphic to a closed unit disk over K with parameter $T_r := p^{-r}T$. Moreover, the residue field κ_r is the function field of the canonical reduction $\bar{D}[r]$ of the affinoid $D[r]$. In fact, $\bar{D}[r]$ is isomorphic to the affine line over k with function field $\kappa_r = k(t)$, where t is the image of T_r in κ_r. For a closed point $\bar{x} \in \bar{D}[r]$, we let $\mathrm{ord}_{\bar{x}}\colon \kappa_r^\times \to \mathbb{Z}$ denote the normalized discrete valuation corresponding to the specialization of $\bar{x}$ on $\bar{D}[r]$. We let ord_∞ denote the unique normalized discrete valuation on κ_r corresponding to the "point at infinity."

Since v_r corresponds to the supremum norm on $D[r]$, it corresponds to a type 2 point $x_r \in X^{\mathrm{Berk}}$ and thus, after a possible extension of K, there is a semistable model $X_{R,r}$ of X whose special fiber $\bar{X}_r$ has a genus 0 component $\bar{V}_r$ corresponding to x_r, connected to the rest of $\bar{X}_r$ at one point (the point at infinity). The intersection of this component with the smooth locus of $\bar{X}_r$ is canonically identified with $\bar{D}[r]$. Thus, κ_r can be identified with the function field $k(\bar{V}_r)$. If $r = 0$, we simply set $X_{R,0} = X_R$, and we take $\bar{V}_0$ to be the irreducible component of $\bar{X}$ containing $\bar{x}_0$. For more details on the above constructions, see [19, §5.3.3]

Notation 2.12 For $F \in K(X)^\times$ and $r \in \mathbb{Q}_{>0}$, we let $[F]_r$ denote the image of $p^{-v_r(F)}F$ in the residue field κ_r.

2.5.2 The different of Cohen, Temkin, and Trushin in disk context

Above, we constructed a type 2 point $x_r \in X^{\mathrm{Berk}}$ for each $r \in \mathbb{Q}_{>0}$, corresponding to $D[r]$. We interpolate type 3 points $x_r \in X^{\mathrm{Berk}}$ for $r \in (\mathbb{R}\backslash\mathbb{Q})_{\geq 0}$ in the obvious way. We define the function $\delta^{\mathrm{Berk}}_{Y/X}\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by $\delta^{\mathrm{Berk}}_{Y/X}(r) := \delta^{\mathrm{Berk}}_{Y/X,x_r}$, and extend it to 0 by continuity. We will write $\delta^{\mathrm{Berk}}(r)$ instead if Y/X is understood.

2.5.3 Kato's Swan conductor in disk context

Suppose $f\colon Y \to X$ is G-Galois with G a finite p-group, and χ is a character of G. Let $r \in \mathbb{Q}_{\geq 0}$ and use the notation of Sect. 2.5.1.

If (after a possible finite extension of K) the normalization $Y_{R,r}$ of $X_{R,r}$ in $K(Y)$ has reduced special fiber $\bar{Y}_r$ with a component $\bar{W}_r$ lying above $\bar{V}_r$, then if $k(\bar{W}_r)/k(\bar{V}_r)$ is purely inseparable we say that f is *residually purely inseparable at* r and if $k(\bar{W}_r)/k(\bar{V}_r)$ is separable, we say f is *residually separable at* r. In either of these cases, Sect. 2.2 gives us a depth Swan conductor $\delta_{\bar{V}_r}(\chi) \in \mathbb{Q}_{>0}$. If χ has degree 1 and f is residually purely inseparable at r, we also get a differential Swan conductor $\omega_{\bar{V}_r}(\chi) \in \Omega^1_{\kappa_r}$. In particular, if f is residually purely inseparable or residually separable at all rational r in some interval of rational numbers $J \subseteq \mathbb{Q}_{>0}$, and $I \subseteq J$ is such that f is residually purely inseparable at all rational $r \in I$, then we have functions

$$\delta_\chi \colon J \to \mathbb{Q}_{\geq 0}$$

and

$$\omega_\chi \colon I \to \Omega^1_{k(t)}$$

given by $\delta_\chi(r) := \delta_{\bar{V}_r}(\chi)$ and $\omega_\chi(r) := \omega_{\bar{V}_r}(\chi)$ (cf. [19, §5.3]). If χ is a degree 1 character of a cyclic group, then δ_χ extends by continuity to a piecewise linear function $\bar{J} \to \mathbb{R}_{\geq 0}$, where $\bar{J}$ is the closure of J in $\mathbb{R}$, with kinks appearing only at rational numbers [19, Proposition 5.10].

The slopes of δ_χ are determined by the orders of zeroes and poles of ω_χ:

Proposition 2.13 ([19, Corollary 5.11]) *If χ is a degree 1 character, $r \geq 0$, and $\delta_\chi(r) > 0$, then the left and right derivatives of δ_χ at r are given by* $\mathrm{ord}_\infty(\omega_\chi(r)) + 1$ *and* $-\mathrm{ord}_0(\omega_\chi(r)) - 1$*, respectively.*

2.5.4 Kato's local vanishing cycles formula in disk context

Assume that $f\colon Y \to X$ is a G-Galois cover with $G \cong \mathbb{Z}/p$. Let $r \in \mathbb{Q}_{\geq 0}$ and use the notation of Sects. 2.5.1 and 2.4.

As mentioned in Sect. 2.5.1, the special fiber $\bar{X}_r$ of $X_{R,r}$ has a component $\bar{V}_r$ that is attached to the rest of $\bar{X}_r$ at one point. If we blow down all the other components, we obtain a model $X'_{R,r}$ of X that has integral and unibranched special fiber $\bar{X}'_r$. Write $Y'_{R,r}$ for the model of Y obtained by normalizing $X'_{R,r}$ in $K(Y)$, and assume the special fiber $\bar{Y}'_r$ is integral and $k(\bar{Y}'_r)/k(\bar{X}'_r)$ is purely inseparable (this may require a finite extension of K).

Note that $\bar{X}'_r$ has an open set canonically identified with $\bar{D}[r]$ whose complement consists of one point, which we will call $\tilde{\infty}$. Let $\delta_{\tilde{\infty}}$ be as in (2.6). Since $X'_{R,r}$ is flat, we have $\delta_{\tilde{\infty}} = g_X$, the genus of X [13, IV, Ex. 1.8]. In this situation, Proposition 2.7 becomes:

Proposition 2.14 *With the notation introduced above, for $\bar{x} \neq \tilde{\infty}$ in $\bar{X}'_r$ and $\bar{y} \in \bar{Y}'_r$ above $\bar{x}$, we have*

$$\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_\chi(r)) = \frac{2\delta_{\bar{y}}}{p-1} - |\mathbb{B} \cap U(\bar{x})|.$$

If $\bar{y} \in \bar{Y}'_r$ lies above $\tilde{\infty}$, we have

$$\mathrm{ord}_{\tilde{\infty}}(\omega_\chi(r)) = \frac{2\delta_{\bar{y}}}{p-1} - \frac{2pg_X}{p-1} - |\mathbb{B} \cap U(\tilde{\infty})|.$$

In particular, $\mathrm{ord}_{\tilde{\infty}}(\omega_\chi(r)) \geq -2pg_X/(p-1) - |\mathbb{B} \cap U(\tilde{\infty})|$.

Lemma 2.15 *Let $r \in \mathbb{Q}_{>0}$. The inverse image of $D[r]$ in Y^{an} is a closed disk iff $\delta_{\bar{y}} = 0$ for all $\bar{y} \in \bar{Y}'_r$ lying over $\bar{D}[r]$. Furthermore, for $r \geq 0$, the inverse image of $D(r)$ in Y^{an} is an open disk iff there exists a decreasing sequence $r_1, r_2, \ldots$ with limit r such that for all r_i, the inverse image of $D[r_i]$ is a closed disk.*

Proof (Compare [1, Lemma 3.10(ii)]) Assume $r > 0$. The inverse image $C[r] := f^{-1}(D[r])$ is an affinoid subdomain of Y. Its canonical reduction $\bar{C}[r]$ may be identified with the inverse image of $\bar{D}[r]$ in $\bar{Y}'_r$. It follows from our assumptions that the map $\bar{Y}'_r \to \bar{X}'_r$ is finite, surjective, and radicial, and hence a homeomorphism on the underlying topological spaces. In particular, $\bar{Y}'_r$ is an irreducible curve over k, with geometric genus 0. Its open subset $\bar{C}[r]$ is the complement of the unique point $\tilde{\infty}' \in \bar{Y}'_r$ lying over $\tilde{\infty}$. It follows that $\bar{C}[r]$ is smooth over k if and only if it is isomorphic to the affine line. The former is equivalent to $\delta_{\bar{y}} = 0$ for all $\bar{y} \in \bar{C}[r]$, and the latter is equivalent to $C[r]$ being a closed disk. This proves the first assertion of the lemma.

Maintain the assumption $r > 0$. The inverse image $C(r) := f^{-1}(D(r))$ is the residue class inside $C[r]$ of a closed point $\bar{y}_0 \in \bar{C}[r]$. By [1, Proposition 3.4], $C(r)$ is an open disk if and only if $\bar{y}_0$ is a smooth point of $\bar{C}[r]$, i.e., if and only if $\delta_{\bar{y}_0} = 0$. On the other hand, it follows from [4, Lemma 2.4], that there exists $\epsilon > 0$ such that $C(r)\backslash C[r']$ is an open annulus for all r' in the interval $(r, r+\epsilon)$. Let $r' \in (r, r+\epsilon)$ be arbitrary. We claim that $C(r)$ is an open disk if and only if $C[r']$ is a closed disk. Clearly, this claim proves the second assertion of the lemma.

To prove the claim, we consider the modification $Y_R'' \to Y_{R,r}'$ corresponding to $C[r'] \subset C(r)$. By this, we mean that the modification is an isomorphism away from $\bar{y}_0$, the exceptional divisor Z is an irreducible and reduced curve which meets the strict transform of $\bar{Y}_r'$ in a unique point $\bar{z}$, and such that $C[r'] =]Z\backslash\{\bar{z}\}[_{Y_R''}$, that is, the subspace of the generic fiber of Y_R'' specializing to $Z\backslash\{\bar{z}\}$. Moreover, we may identify the canonical reduction $\bar{C}[r']$ of $C[r']$ with $Z - \{\bar{z}\}$. But then $C(r)\backslash C[r'] =]\bar{z}[_{Y_R''}$. By our choice of r', we know that $C(r)\backslash C[r']$ is an open annulus. It follows that z is an ordinary double point of $\bar{Y}''$ [1, Proposition 3.4]. Using [1, p. 8, (2)], we see that $\delta_{\bar{y}_0} = 0$ if and only if $\bar{C}[r'] = Z - \{\bar{z}\}$ is smooth of genus zero. By the same argument as above, this is equivalent to $C[r']$ being a closed disk. Now the proof of the lemma is complete for $r > 0$.

For $r = 0$, the same argument works, replacing $\bar{D}[r]$ with $\bar{V}_0$ from Sect. 2.5.1. □

2.6 Disks and slopes

Maintain the notation of Sect. 2.5. Assume there is an interval of rational numbers $J \subseteq \mathbb{Q}_{>0}$ such that for all rational r in J, either f is purely inseparable at r or f is residually separable at r. Then, if f is Galois, we can define δ_χ on the closure $\bar{J}$ of J in $\mathbb{R}$ and ω_χ on the subset I of J where f is purely inseparable. Recall from Sect. 2.5.2 that, whether or not f is Galois, we define the function $\delta_{Y/X}^{\mathrm{Berk}} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\delta_{Y/X}^{\mathrm{Berk}}(r) = \delta_{Y/X,x_r}^{\mathrm{Berk}}$, where $x_r \in X^{\mathrm{Berk}}$ is the point corresponding to $D[r]$. Let $\mathbb{B}(r)$ (resp. $\mathbb{B}[r]$) be the subset of the branch locus of f lying in $D(r)$ (resp. $D[r]$).

Lemma 2.16 *Suppose $G = \mathbb{Z}/p$ and $f\colon Y \to X$ is a G-Galois cover. Let $r \in \mathbb{Q}_{>0}$, and assume that $\mathbb{B}(r)$ is non-empty. Then $(f^{\mathrm{an}})^{-1}(D(r))$ is connected.*

Proof By assumption, the restriction of f above $D(r)$ is a ramified cover and thus clearly connected. □

Lemma 2.17 *Suppose $G = \mathbb{Z}/p$ with χ a faithful degree 1 character of G, and $f\colon Y \to X$ is a G-Galois cover. Let $r \in \mathbb{Q}_{>0}$, and assume that $\delta_\chi(r) = 0$ and that either $(f^{\mathrm{an}})^{-1}(D[r])$ is a closed disk or (f^{an}) has no branch points in $D(r)$. Then $(f^{\mathrm{an}})^{-1}(D(r))$ is the disjoint union of p open disks.*

Proof The inverse image $C[r] := (f^{\mathrm{an}})^{-1}(D[r])$ is an affinoid subdomain of Y. The map $C[r] \to D[r]$ induces a finite and flat morphism of degree p between the canonical reductions, $\bar{C}[r] \to \bar{D}[r]$, which are affine curves over k and $\bar{D}[r]$ is an affine line. If $C[r]$ is a closed disk, then $\bar{C}[r]$ is an affine line as well, and the map $\bar{C}[r] \to \bar{D}[r]$ is generically étale. Now the Riemann–Hurwitz formula shows that this map is actually étale (it is totally and wildly ramified over the unique point at infinity). But $D(r) \subset D[r]$ is the residue class of a closed point $\bar{x} \in \bar{D}[r]$. It follows that $f^{-1}(D(r))$ splits into p copies of $D(r)$.

If, on the other hand, (f^{an}) has no branch points in $D(r)$, then by purity of the branch locus, the map $\bar{C}[r] \to \bar{D}[r]$ is étale above $\bar{x}$. We conclude as above. □

Corollary 2.18 *Suppose $G = \mathbb{Z}/p$ and χ is a faithful degree 1 character. Let $f\colon Y \to X$ be a G-Galois cover. Let $r \in \mathbb{Q}_{>0}$, and assume that $(f^{\mathrm{an}})^{-1}(D(r))$ is connected. The following are equivalent:*

(i) *$(f^{\mathrm{an}})^{-1}(D[r])$ is a closed disk.*
(ii) *f is residually purely inseparable at r and $\mathrm{ord}_\infty(\omega_\chi(r)) = |\mathbb{B}[r]| - 2$.*
(iii) *$\delta_\chi(r)$ has left-slope $|\mathbb{B}[r]| - 1$ at r.*
(iv) *$\delta^{\mathrm{Berk}}_{Y/X}$ has left-slope $(p-1)(|\mathbb{B}[r]| - 1)/p$ at r.*

Furthermore, it is always the case that $\delta^{\mathrm{Berk}}_{Y/X}$ (resp. $\delta_\chi(r)$) has left-slope at most $(p-1)(|\mathbb{B}[r]| - 1)/p$ (resp. $|\mathbb{B}[r]| - 1$) at r.

Proof That (ii) implies (i) follows from Lemma 2.15, Proposition 2.14 applied to all $\bar{x} \neq \bar{\infty}$, and the fact that a differential on $\mathbb{P}^1$ has total degree 2. The reverse implication follows by the same argument, combined with Lemma 2.17.

That (ii) implies (iii) follows from Proposition 2.13. If f is residually purely inseparable at r, the same proposition shows that (iii) implies (ii). Suppose f is residually separable at r. Then the left-slope of $\delta_\chi(r)$ is non-positive. Thus (iii) holds only if $|\mathbb{B}[r]| \leq 1$. By Lemma 2.17, we cannot have $|\mathbb{B}[r]| = 0$. By Corollary 3.15 below (which does not depend on this corollary), we cannot have $|\mathbb{B}[r]| = 1$. Thus (iii) implies (ii) in all cases.

The equivalence of (iii) and (iv) follows from Lemma 2.3.

For the last assertion, note that Proposition 2.14 shows that $\mathrm{ord}_\infty(\omega_F(r))$ is at most $|\mathbb{B}[r]| - 2$. If f is residually purely inseparable at r, the proof that (ii) is equivalent to (iii) now carries through exactly. If f is residually separable at r, we know from the argument above that $|\mathbb{B}[r]| > 1$, in which case the last assertion is automatic. □

In order to generalize Corollary 2.18 to general p-groups, we need a result about canonical metrics and multiplicities on Berkovich spaces. Recall that Berkovich curves come with a canonical metric on their type 2 and 3 points (see, e.g., [7, §5]). For $s,\ s' > 0$, the definition of this metric shows that the path from x_s to $x_{s'}$ has length $|s' - s|$. Suppose $s, s' \in I$. Since f is purely inseparable on I, we have that $(f^{\mathrm{Berk}})^{-1}(x_r)$ has exactly one preimage for each $r \in [s, s']$. So f^{Berk} has multiplicity $\deg f$ above the interval $A := [x_s, x_{s'}] \subseteq X^{\mathrm{Berk}}$. It is a consequence of [9, Lemma 3.5.8], that the restriction of f^{Berk} to the interval $(f^{\mathrm{Berk}})^{-1}(A)$ is linear and expands distances by a factor of $\deg f$. Note that, after removing finitely many type 2 points corresponding to higher-genus curves, $f^{-1}(A)$ is a union of skeletons of annuli.

Suppose $\varphi\colon Z \to X$ is an intermediate cover between Y and X (with $Z \neq Y$). If $z_r \in Z^{\mathrm{Berk}}$ is the point lying above x_r, then z_r corresponds to a component $\bar{W}_r$ of the special fiber of some semistable model Z_R of Z. Again, there are depth and differential Swan conductors $\delta_{Y/Z,\bar{W}_r}(\chi)$ for $r \in J$ and $\omega_{Y/Z,\bar{W}_r}(\chi)$ for $r \in I$. The function $\delta_{Y/Z,\bar{W}_r}(\chi)$, written as $\delta_{Y/Z,\chi}$ when thought of as a function of r, extends to a piecewise linear function from $\bar{J}$ to $\mathbb{R}_{\geq 0}$, just as δ_χ does [21, Proposition 2.3.35]. Alternatively, we can think of $\delta_{Y/Z,\bar{W}_r}(\chi)$ as giving a function on the interval $B := \{z_r \in Z^{\mathrm{Berk}} \mid r \in \bar{J}\}$. At any particular

r, the function $\delta_{Y/Z,\bar{W}_r}(\chi)$ has left and right slopes with respect to r, as well as with respect to the canonical metric on B.

Proposition 2.19 *Let $\varphi\colon Z \to X$ be as above. Let $r \in J$. If $\bar{\infty}_{\bar{Z}} \in \bar{W}_r$ is the unique point above $\bar{\infty} \in \bar{X}$, then the left-slope of $\delta_{Y/Z,\bar{W}_r}(\chi)$ at r, thought of as a function of r, is $(\mathrm{ord}_{\bar{\infty}_{\bar{Z}}}(\omega_{Y/Z,\bar{W}_r}(\chi)) + 1)/\deg\varphi$.*

Proof There exists $\epsilon > 0$ such that the interval $(z_{r-\epsilon}, z_r)$ is the skeleton of an open annulus. Then the left-slope of $\delta_{Y/Z,\bar{W}_r}(\chi)$ relative to the canonical metric on B is $\mathrm{ord}_{\bar{\infty}_{\bar{Z}}}(\omega_{Y/Z,\bar{W}_r}(\chi)) + 1$ by Proposition 2.13. We divide this slope by $\deg\varphi$ to get the left-slope relative to the canonical metric on the interval $(x_{r-\epsilon}, x_r)$, because φ^{Berk} expands distances by a factor of $\deg\varphi$. This is the left-slope with respect to r. □

We now give the generalization of Corollary 2.18.

Corollary 2.20 *Suppose $f\colon Y \to X$ is a composition of finitely many $\mathbb{Z}/p$-Galois covers. Let $r \in \mathbb{Q}_{>0}$, and assume that f is residually purely inseparable at r.*

(i) *There exists $m_{\mathrm{diff}}(r) \in \mathbb{Q}$, depending only on the number of branch points in each $\mathbb{Z}/p$-subquotient cover of f in $D[r]$, such that $\delta^{\mathrm{Berk}}_{Y/X}$ has left-slope $\leq m_{\mathrm{diff}}$ at r, with equality holding iff $(f^{\mathrm{an}})^{-1}(D[r])$ is a closed disk.*
(ii) *Furthermore, if f is G-Galois and if χ is an irreducible, faithful character on G, then there exists $m_{\mathrm{Swan}}(r) \in \mathbb{Q}$, depending only on the number of branch points in each $\mathbb{Z}/p$-subquotient cover of f in $D[r]$, such that $\delta_{Y/X,\chi}$ has left-slope $\leq m_{\mathrm{Swan}}$ at r and such that if $(f^{\mathrm{an}})^{-1}(D[r])$ is a closed disk, then equality holds.*
(iii) *In the situation of (ii), suppose $H \subseteq G$ is a cyclic subgroup such that χ is induced from a character of H, with $H' \subseteq H$ the unique subgroup of order p. Let $\varphi\colon Y/H' \to X$ be the quotient morphism of f and suppose $(\varphi^{\mathrm{an}})^{-1}(D[r])$ is a closed disk. Then $\delta_{Y/X,\chi}$ having left-slope $m_{\mathrm{Swan}}(r)$ implies that $(f^{\mathrm{an}})^{-1}(D[r])$ is a closed disk.*
(iv) *If G is cyclic, and χ is an irreducible, faithful character on G, then we can take $m_{\mathrm{Swan}}(r) = |\mathbb{B}[r]| - 1$, where $\mathbb{B}[r]$ is the set of branch points of f in $D[r]$.*

Proof Let $Y =: Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 := X$ be a composition series of $\mathbb{Z}/p$-covers for f. If $x_r \in X^{\mathrm{Berk}}$ is the point corresponding to $D[r]$, let $\delta^{\mathrm{Berk}}_i(r)$, $1 \leq i \leq n$ be the different of $Y^{\mathrm{Berk}}_i \to Y^{\mathrm{Berk}}_{i-1}$ at the point above x_r. By Proposition 2.1, $\delta^{\mathrm{Berk}}_{Y/X}(r) = \sum_{i=1}^n \delta^{\mathrm{Berk}}_i(r)$.

Let $\mathbb{B}_i[r]$ be the set of branch points of $Y_i \to Y_{i-1}$ lying above $D[r]$. For $0 \leq i \leq n$, let $x_{r,i}$ be the unique point of Y^{Berk}_i lying above x_r, and let $\bar{W}_{r,i}$ be the corresponding irreducible component of a semistable model of Y_i. For $\bar{x} \in \bar{W}_{r,i}$, use the notation $\delta_{\bar{x}}$ as in (2.6). Let $\bar{\infty}_{\bar{Y}_{i-1}}$ be as in Proposition 2.19. If ψ is a faithful irreducible character of $\mathbb{Z}/p$, we have from (2.9) that, for $0 \leq i \leq n-1$,

$$\begin{aligned}
&\mathrm{ord}_{\infty_{\bar{Y}_{i-1}}}(\omega_{Y_i/Y_{i-1},\bar{W}_{r,i}}(\psi)) + 1 \\
&\quad = |\mathbb{B}_i[r]| - 1 + \sum_{\bar{x} \in \bar{W}_{r,i-1} \setminus \infty_{\bar{Y}_{i-1}}} \frac{2p}{p-1}\delta_{\bar{x}} - \sum_{\bar{y} \in \bar{W}_{r,i} \setminus \infty_{\bar{Y}_i}} \frac{2}{p-1}\delta_{\bar{y}}.
\end{aligned}$$

Thus, by Proposition 2.19, we have that the left-slope of $\delta_{Y_i/Y_{i-1},\bar{W}_{r,i}}(\psi)$ as a function of r is

$$\frac{1}{p^{i-1}}\left(|\mathbb{B}_i[r]|-1+\sum_{\bar{x}\in\bar{W}_{r,i-1}\backslash\infty_{\bar{Y}_{i-1}}}\frac{2p}{p-1}\delta_{\bar{x}}-\sum_{\bar{y}\in\bar{W}_{r,i}\backslash\infty_{\bar{Y}_i}}\frac{2}{p-1}\delta_{\bar{y}}\right). \tag{2.21}$$

Combining this with Lemma 2.3 and Proposition 2.1, and noting that $\delta_{\bar{x}} = 0$ for all $\bar{x}\in\bar{X}\backslash\bar{\infty}$, we get that the left-slope of $\delta^{\mathrm{Berk}}_{Y/X}$ at r is

$$\sum_{i=1}^{n}\frac{(p-1)(|\mathbb{B}_i[r]|-1)}{p^i}-\sum_{\bar{y}\in\bar{W}_{r,n}\backslash\infty_{\bar{Y}_n}}\frac{2\delta_{\bar{y}}}{p^{n-1}}.$$

Taking

$$m_{\mathrm{diff}}(r)=\sum_{i=1}^{n}\frac{(p-1)(|\mathbb{B}_i[r]|-1)}{p^i}$$

and using Lemma 2.15 proves (i).

For (ii) and (iii), we first assume $G=\mathbb{Z}/p$. Then (ii) and (iii) follow from (i) and Lemma 2.3, taking

$$m_{\mathrm{Swan}}(r)=\frac{pm_{\mathrm{diff}}(r)}{p-1}$$

(note that the condition in (iii) always holds in this case).

Now, assume $G\neq\mathbb{Z}/p$. We take the composition series $Y =: Y_n\to Y_{n-1}\to\cdots\to Y_1\to Y_0 := X$ to be such that there exists m with $Y_m = Y/H$. Combining the formula for $\delta_{Y_i/Y_{i-1},\bar{W}_{r,i-1}}(\psi)$ in (2.21) with Lemma 2.3 and Proposition 2.5, we get that the left-slope of $\delta_{Y/X,\chi}$ is

$$p^m\left(\sum_{i=1}^{n-1}\frac{(p-1)(|\mathbb{B}_i[r]|-1)}{p^i}+\frac{|\mathbb{B}_n[r]|-1}{p^{n-1}}-\sum_{\bar{y}\in\bar{W}_{r,n}\backslash\infty_{\bar{Y}_n}}\frac{2\delta_{\bar{y}}}{p^{n-1}(p-1)}\right.$$
$$\left.+\sum_{\bar{y}\in\bar{W}_{r,n-1}\backslash\infty_{\bar{Y}_{n-1}}}\frac{2\delta_{\bar{y}}}{p^{n-1}(p-1)}\right).$$

Taking

$$m_{\mathrm{Swan}}(r)=p^m\left(\sum_{i=1}^{n-1}\frac{(p-1)(|\mathbb{B}_i[r]|-1)}{p^i}+\frac{|\mathbb{B}_n[r]|-1}{p^{n-1}}\right) \tag{2.22}$$

and using Lemma 2.15 proves (ii) and (iii) (note that Lemma 2.15 shows that the assumption of (iii) is satisfied exactly when $\delta_{\bar{y}}=0$ for all $\bar{y}\in\bar{W}_{r,n-1}\backslash\infty_{\bar{Y}_{n-1}}$).

If G is cyclic, we have $G = H$ and $m = 0$. Then (2.22) simplifies to

$$m_{\mathrm{Swan}}=\sum_{i=1}^{n-1}\frac{(p-1)|\mathbb{B}_i[r]|}{p^i}+\frac{|\mathbb{B}_n[r]|}{p^{n-1}}-1.$$

For $1\leq s\leq n$, let $\mathbb{B}^s[r]$ be the set of branch points of f in $D[r]$ with branching index p^{n-s+1}. Then $|\mathbb{B}_i[r]|=\sum_{j=1}^{i}|\mathbb{B}^j[r]|p^{j-1}$. Plugging this into the equation for $m_{\mathrm{Swan}}(r)$ and simplifying, we obtain $m_{\mathrm{Swan}}(r)=\left(\sum_{i=1}^{n}|\mathbb{B}^i[r]|\right)-1$. This proves (iv). □

Remark 2.23 Equation (2.8) in Proposition 2.7 follows from (2.9) using a similar argument as in the proof of Corollary 2.20 above. Using the notation of Corollary 2.20 for G a cyclic group, one adds the equations (2.9) for Y_i/Y_{i-1} together for $1 \leq i \leq n$, multiplied by $(p-1)/p^i$ for $i < n$ and by $1/p^{n-1}$ for $i = n$ (just like in the proof of (ii) and (iii) above). The left-hand side becomes $\mathrm{ord}_{q_X^{-1}(\bar{x})}(\omega_{\bar{V}}(\chi))$, and the right-hand side becomes

$$\frac{2}{p^{n-1}(p-1)}(\delta_{\bar{y}} - \delta_{\bar{z}}) - 2\delta_{\bar{x}} - \sum_{i=1}^{n-1} \frac{(p-1)(|\mathbb{B}_i \cap U(\bar{x})|)}{p^i} + \frac{|\mathbb{B}_n \cap U(\bar{x})|}{p^{n-1}},$$

which is equal to the right-hand side of (2.8) as in the proof of (iv) above.

3 Individual $\mathbb{Z}/p$-covers

In this section, we do an in-depth analysis on individual $\mathbb{Z}/p$-covers, expanding on the analysis that was done in [19, §5.4, 5.5]. Furthermore, we correct an error that was present in that paper (see Remark 3.13). Throughout, we maintain the notation and assumptions of Sect. 2.5. In particular, $f: Y \to X$ is an $\mathbb{Z}/p$-cover of smooth, projective curves over K, and D is an open unit disk inside X^{an}. For $r \in \mathbb{Q}_{\geq 0}$, we have $D[r]$ and $D(r)$ as in Sect. 2.5. If χ is a character of $\mathbb{Z}/p$, we have functions δ_χ and ω_χ defined on the appropriate intervals as in Sect. 2.5. Furthermore, we assume that $\zeta_p \in K$.

3.1 Explicit formulas for a $\mathbb{Z}/p$-cover

We recall a result from [19] that will be our main computational tool. Suppose that χ is any *faithful* degree 1 character of $\mathbb{Z}/p$. Using Kummer theory, there exists $F \in K(X)^\times$ such that $\sigma(F)/F = \chi(\sigma)$ for all $\sigma \in \mathbb{Z}/p$. For any such F, we write δ_F and ω_F for the functions δ_χ and ω_χ from Sect. 2.5.3.

The following proposition is contained in [19, Proposition 5.17].

Proposition 3.1 *Let F be as above and $r \in \mathbb{Q}_{>0}$. Suppose that $v_r(F) = 0$, that $H \in K(X)$, and that $g := [F - H^p]_r \notin \kappa_r^p$. Suppose, moreover, that $K(Y)/K(X)$ is weakly unramified with respect to v_r (which is always the case if K is chosen large enough).*

(i) *We have*

$$\delta_F(r) = \max\left(\frac{p}{p-1} - v_r(F - H^p),\ 0\right).$$

(ii) *If $\delta_F(r) > 0$, then*

$$\omega_F(r) = \begin{cases} dg/g & \text{if } \delta_F(r) = p/(p-1), \\ dg & \text{if } 0 < \delta_F(r) < p/(p-1). \end{cases}$$

If there is no H such that $g \notin \kappa_r^p$, then $\delta_F(r) = 0$.

Remark 3.2 It is not difficult to see that replacing F with F^m for m prime to p (which is equivalent to replacing χ with the faithful degree 1 character χ^m) does not affect δ_F and multiplies ω_F by the scalar m.

The following proposition is contained in [19, Proposition 5.9].

Proposition 3.3 *Let F_1 and F_2 be as above and $r \in \mathbb{Q}_{>0}$. Write $\delta_1, \delta_2, \omega_1, \omega_2$ for $\delta_{F_1}(r)$, $\delta_{F_2}(r)$, $\omega_{F_1}(r)$, and $\omega_{F_2}(r)$, respectively. Write δ_3, ω_3 for $\delta_{F_1F_2}(r)$, $\omega_{F_1F_2}(r)$, respectively. Then $\delta_3 \leq \max(\delta_1, \delta_2)$. If $\omega_1 + \omega_2 \neq 0$, then $\omega_3 = \omega_1 + \omega_2$ and $\delta_3 = \max(\delta_1, \delta_2)$.*

Definition 3.4 We call $F \in K(X)^\times$ a *Kummer representative* for the $\mathbb{Z}/p$-cover $f\colon Y \to X$ if there exists a faithful degree 1 character χ of $\mathbb{Z}/p$ such that $\sigma(F)/F = \chi(\sigma)$ for all $\sigma \in \mathbb{Z}/p$. We make the same definition for $\mathbb{Z}/q$-covers for any prime $q \neq p$ (this will be needed in Sect. 4.2).

Thus, a Kummer representative for f is any element ϕ of $K(X)^\times$ such that $K(Y) \cong K(X)[\sqrt[p]{\phi}]$. In light of Remark 3.2, the following definition makes sense.

Definition 3.5 (*cf.* [19, Proposition 5.20]) If $f\colon Y \to X$ is a $\mathbb{Z}/p$-cover with $F \in K(X)$ as Kummer representative, m is an integer, and $r_0 \in \mathbb{Q}_{>0}$, then we define $\lambda_{m,r_0}(f)$ to be the maximum of all $r \in (0, r_0]$ such that the left-slope of δ_F at r is (strictly) less than m, or 0 if there is no such r.

Remark 3.6 Let $m_{\mathrm{Swan}}(r)$ be as in Corollary 2.20(ii). If $m = m_{\mathrm{Swan}}(r)$ for all $r \in (0, r_0]$ and f is residually purely inseparable at all these r, then Corollary 2.20(ii) allows us to replace "strictly less than m" by "not equal to m" in Definition 3.5.

The following proposition will be useful later, when we need to distinguish cases based on whether $p | m_{\mathrm{Swan}}(r)$.

Proposition 3.7 *If $\delta_F(r) \neq 0$ and the left-slope or right-slope of δ_F at r is divisible by p, then $\delta_F(r) = p/(p-1)$, and the left-slope or right-slope in question is in fact* 0.

Proof If $0 < \delta_F(r) < p/(p-1)$, then $\omega_F(r)$ is exact and thus never has order congruent to $-1 \pmod{p}$. Using Proposition 2.13, this contradicts having slope divisible by p. The last statement follows since δ_F is piecewise linear. $\square$

3.2 Kummer representatives of $\mathbb{Z}/p$-covers

Maintain the notation of Sect. 2.5. In this section, we fix $r_0 \in \mathbb{Q}_{>0}$, and we assume that the $\mathbb{Z}/p$-cover $f\colon Y \to X$ has no branch points in $D \backslash D[r_0]$.

Lemma 3.8 *Suppose $f\colon Y \to X$ is as above, and pick K large enough so that the nonzero branch points $x_1, \ldots, x_n$ of f inside D are defined over K (we think of $x_1, \ldots, x_n$ as elements of valuation $\geq r_0$ in R). Then there exists $F = U\tilde{F} \in K(X)$ such that $K(Y) = K(X)[\sqrt[p]{F}]$, that*

$$\tilde{F} = T^{\alpha_0} \prod_{i=1}^{N} \left(1 - x_i T^{-1}\right)^{\alpha_i} \tag{3.9}$$

for some $\alpha_i \in \{1, \ldots, p-1\}$ for $i > 0$, and $\alpha_0 \in \{0, \ldots, p-1\}$, and that U is a unit on D.

Proof If F is such that $K(Y) = K(X)[\sqrt[p]{F}]$, then F has poles/zeroes of prime-to-p order exactly at the branch points of f. Since $T \in K(X)$ and F can be chosen up to multiplication by pth powers, the lemma follows. $\square$

Remark 3.10 With the α_i chosen as above, $\tilde{F}$ is, in fact, a Laurent polynomial, but it will be convenient of us to think of it as a power series.

Remark 3.11 Let $S = R[[T]] \otimes_R K$. Since U is a unit on D and is contained in $K(X)$, we have $U \in S^\times$. In particular, after a finite extension of K and possibly multiplying U by a pth power, we may write $U = 1 + \sum_{i=1}^{\infty} b_i T^i$ with $\nu(b_i) \geq 0$ for all i.

Remark 3.12 Note that, if $\tilde{F}$ from Lemma 3.8 is expanded out as a power series, we have $\tilde{F} = T^{\alpha_0}(1 + \sum_{i=1}^{\infty} a_i T^{-i})$, with $\nu(a_i) \geq r_0 i$.

Remark 3.13 In [19, p. 249], it was incorrectly claimed, under an assumption equivalent to $\alpha_0 = 0$, that F could be chosen in Lemma 3.8 such that $F = 1 + \sum_{i=1}^{\infty} a_i T^{-i}$ with $\nu(a_i) \geq r_0 i$. This is only true in general if $X = \mathbb{P}^1$ and f has no branch points outside D (as in this case, we can take $U = 1$) (the assumption $X = \mathbb{P}^1$ is not stated at the beginning of §5 of [19], but the results proved in that section are only used for $X = \mathbb{P}^1$). Much of the rest of Sect. 3.2 is meant to adapt [19, Proposition 5.20] to the situation where we do not necessarily assume $U = 1$.

From now on, we will use the notation δ_F, δ_U, $\delta_{\tilde{F}}$, etc. from Sect. 3.1. Note that this all makes sense for $U \in S^{\times}$ and $\tilde{F}$ a power series as in Remark 3.12, even if U and $\tilde{F}$ are not in $K(X)$.

Proposition 3.14 *If $\alpha_0 \neq 0$ in (3.9), then $\delta_F(r) = p/(p-1)$ for all $r \in (0, r_0]$. If $\alpha_0 = 0$ in (3.9), then $\delta_F(r) < p/(p-1)$ for $r \in (0, r_0)$.*

Proof If $\alpha_0 \neq 0$, then there will be a t^{α_0} term in $[\tilde{F}]_r$. Since $p \nmid \alpha_0$, we have $\delta_{\tilde{F}}(r) = p/(p-1)$ by Proposition 3.1(i). Also, $\delta_U(r) < p/(p-1)$ by Proposition 3.1(i). From Proposition 3.3, we conclude that $\delta_F(r) = p/(p-1)$.

If $\alpha_0 = 0$, then $\nu_r(F - 1) > 0$ for $r \in (0, r_0)$. By Proposition 3.1, we get $\delta_F(r) < p/(p-1)$.

Corollary 3.15 *If f has exactly one branch point x_1 in $D[r_0]$, then $\delta_F(r) = p/(p-1)$ for all $r \in (0, r_0]$.*

Proof From (3.9), we must have $\alpha_0 \neq 0$ (otherwise both 0 and x_1 are branch points). Now use Proposition 3.14. ◻

Lemma 3.16 (i) *Suppose $U = 1 + \sum_{i=1}^{\infty} a_i T^i \in 1 + TR[[T]]$, and let $s \in \mathbb{Z}$. After a possible finite extension of K, there exists $I := 1 + \sum_{i=1}^{s} b_i T^i \in R[T]$ such that if $U - I^p = \sum_{i=1}^{\infty} c_{-i} T^i$, then $c_{-p} = c_{-2p} = \cdots c_{-sp} = 0$.*

(ii) *Suppose $\tilde{F} = 1 + \sum_{i=1}^{\infty} a_i T^{-i} \in 1 + T^{-1}R[[T^{-1}]]$ with $\nu(a_i) \geq r_0 i$, and let $s \in \mathbb{Z}$. After a possible finite extension of K, there exists $I := 1 + \sum_{i=1}^{s} b_i T^{-i} \in R[T^{-1}]$ such that if $\tilde{F} - I^p = \sum_{i=1}^{\infty} c_i T^{-i}$, then $c_p = c_{2p} = \cdots c_{sp} = 0$ and $\nu(c_i) \geq r_0$ for all i.*

In both cases, there are only finitely many solutions for the b_i and c_i, and they are given as solutions of polynomial equations in the a_i. In particular, the valuation of the c_i does not depend on which solution is chosen, and if we think of the a_i as indeterminates, the b_i and c_i vary analytically with the a_i.

Proof Part (ii) is just [19, Lemma 5.18 and Remark 5.19], and the proof of (i) is exactly the same. ◻

In the next lemma, for $s \in \mathbb{Z}_{\geq 0}$ and $A \in K[[T]]$ (resp. $K[[T^{-1}]]$), write A_s for the degree s truncation of A (resp. the degree $-s$ truncation). If $s < 0$, simply write $A_s = 1$ (the lemma is vacuous anyway in this case).

Lemma 3.17 (i) *Let $U \in 1 + TR[[T]]$. Let $s \in \mathbb{Z}$ and $r \in \mathbb{Q}_{>0}$. Assume that, if $\delta_U(r) > 0$, then $\mathrm{ord}_{\infty}(\omega_U(r)) \geq -s - 1$. Let $I \in 1 + TR[[T]]$ be such that $U - I^p$ has no terms of degree i for $i \in \{p, 2p, \ldots, sp\}$.*

(a) *If $\nu_r((U - I^p)_s) < p/(p-1)$, then there exists $H \in 1 + TK[[T]]$ such that $[U - H^p]_r \notin \kappa_r^p$. In this case, $\delta_U(r) > 0$, and we have $d[U - H^p]_r = d[(U - I^p)_s]_r$ and $\nu_r(U - H^p) = \nu_r((U - I^p)_s)$.*

(b) *If $\nu_r((U - I^p)_s) \geq p/(p-1)$, then $\delta_U(r) = 0$.*

(ii) *Let $\tilde{F} \in 1 + T^{-1}R[[T^{-1}]]$ with the coefficient of each T^{-i} having valuation at least $r_0 i$. Let $s \in \mathbb{Z}$ and $r \in \mathbb{Q} \cap (0, r_0)$. Assume that, if $\delta_{\tilde{F}}(r) > 0$, then $\mathrm{ord}_{\bar{0}}(\omega_{\tilde{F}}(r)) \geq -s - 1$. Let $I \in 1 + TR[[T]]$ be such that $\tilde{F} - I^p$ has no terms of degree i for $i \in \{p, 2p, \ldots, sp\}$.*

(a) *If $\nu_r((\tilde{F} - I^p)_s) < p/(p-1)$, then there exists $H \in 1 + TK[[T]]$ such that $[\tilde{F} - H^p]_r \notin \kappa_r^p$. In this case, $\delta_F(r) > 0$, and we have $d[\tilde{F} - H^p]_r = d[(\tilde{F} - I^p)_s]_r$ and $\nu_r(\tilde{F} - H^p) = \nu_r((\tilde{F} - I^p)_s)$.*

(b) *If $\nu_r((\tilde{F} - I^p)_s) \geq p/(p-1)$, then $\delta_{\tilde{F}}(r) = 0$.*

Proof We prove (i). The proof of (ii) is exactly the same.

Recall that $T_r = p^{-r}T$. For this proof, we write all power series in terms of T_r. In particular, write $U - I^p = \sum_{i=1}^{\infty} d_{-i} T_r^i$. By assumption, $\nu(d_{-i}) = \nu_r(d_{-i}T_r^i) \geq ir$. Suppose we are in case (a). The first assertion in (a) follows from applying Lemma 3.16 in order to eliminate all terms $d_{-ip}T_r^{ip}$ with $i > 0$ and $\nu_r(d_{-ip}T_r^{ip}) < p/(p-1)$ (there are only finitely many such terms). Proposition 3.1 shows that $\delta_U(r) > 0$.

Since $[U - H^p]_r \notin \kappa_r^p$, we know that $\omega_U(r) = d[U - H^p]_r$. By assumption, $d[U - H^p]_r = \alpha(t)dt$ where $\alpha(t)$ has degree at most $s - 1$. That is, $d[U - H^p]_r = d[(U - H^p)_s]_r$ and $\nu_r(U - H^p) = \nu_r((U - H^p)_s)$.

Now, write $I - H = (\sum_{i=1}^{\infty} a_i T_r^i)$. Let $\beta = \min_{1 \leq i \leq s} \nu(a_i)$ and let $j \in \{1, \ldots, s\}$ be such that $\nu(a_j) = \beta$. Since $\delta_U(r) > 0$, Proposition 3.1 shows that terms of coefficient valuation at least $p/(p-1)$ in $U - H^p$ affect neither $\nu_r(U - H^p)$ nor $[U - H^p]_r$. Thus we may assume that either $(I^p - H^p)_s = 0$ or $\beta < 1/(p-1)$. If $(I^p - H^p)_s = 0$ we are done by the previous paragraph, so assume otherwise. Then $\nu_r((I^p - H^p)_s) = p\beta$, and the only terms of $(I^p - H^p)_s$ that can have coefficient valuation $p\beta$ are those whose degrees are divisible by p. Consequently, $U - H^p = U - I^p + (I^p - H^p)$ includes a term with ν_r equal to $p\beta$ (the T_r^{jp} term), and thus some term of degree not divisible by p with valuation $\leq p\beta$. Thus $\nu_r(U - H^p) \leq p\beta$. This implies that $d[(U - H^p)_s]_r = d[(U - I^p)_s]_r$ and $\nu_r((U - H^p)_s) = \nu_r((U - I^p)_s)$. Combining this with the paragraph above proves the rest of part (a).

For part (b), we argue by contradiction. If $\delta_U(r) > 0$, then there exists $H \in 1 + TK[[T]]$ such that $[U - H^p]_r \notin \kappa^p$ and $\nu_r(U - H^p) < p/(p-1)$. But then we are in the situation of part (a), and by part (a) we have

$$\frac{p}{p-1} > \nu_r(U - H^p) = \nu_r\left((U - I^p)_s\right) \geq \frac{p}{p-1},$$

a contradiction. □

Corollary 3.18 *In the situation of Lemma 3.17, so long as $\delta_U(r) < p/(p-1)$, we have $\delta_U(r) = \delta_{(U-I^p)_s}(r)$ and $\omega_U(r) = \omega_{(U-I^p)_s}(r)$. The same holds for $\tilde{F}$.*

Proof Immediate from Lemma 3.17 and Proposition 3.1. □

3.3 A function on power series

Suppose $\tilde{F} \in 1 + T^{-1}R[[T^{-1}]]$ and $U \in 1 + TR[[T]]$. Suppose further that $r_0 \in \mathbb{Q}_{>0}$ and $v_{r_0}(F-1) \geq 0$ (this is the case when $\tilde{F}$ is as in (3.9) and $\alpha_0 = 0$). Let $m \in \mathbb{Z}$ be prime to p. Pick $s_U \in \mathbb{Z}_{>0}$ such that $-s_U < m$. Let $I_U, I_{\tilde{F}}$ be the I guaranteed by Lemma 3.16(i) and (ii) for $s = s_U, m$, respectively. Write

$$U - I_U^p = \sum_{i=1}^{\infty} c_{-i} T^i, \tag{3.19}$$

and

$$\tilde{F} - I_{\tilde{F}}^p = \sum_{i=1}^{\infty} c_i T^{-i}, \tag{3.20}$$

and recall that the c_i vary analytically with the coefficients of U and F by Lemma 3.16.

Definition 3.21 (*cf.* [19, Proposition 5.20]) Given U, $\tilde{F}$, and s_U as above, then

$$\mu_{s_U,m}(U, \tilde{F}) := \max\left(\left\{\frac{v(c_m) - v(c_i)}{m - i} \mid -s_U \leq i < m,\ p \nmid i\right\} \cup \{0\}\right)$$

as long as $c_m \neq 0$. If $c_m = 0$, we set $\mu_{s_U,m}(U, \tilde{F}) = \infty$.

3.4 Swan conductor kinks

Recall that, for $m \in \mathbb{Z}$, for $r_0 \in \mathbb{Q}_{>0}$, and for $f\colon Y \to X$ a $\mathbb{Z}/p$-cover as in Sect. 2.5, we defined λ_{m,r_0} in Definition 3.5. Assume that f has no branch points in $D\backslash D[r_0]$. For $r \in \mathbb{Q}_{>0}$, let $\mathbb{B}[r]$ be the number of branch points of f in $D[r]$. The following proposition relates the functions λ_{m,r_0} and $\mu_{s_U,m}$.

Proposition 3.22 (cf. [19, Proposition 5.20]) *Suppose $F \in K(X)$ is a Kummer representative for the $\mathbb{Z}/p$-cover f and that $F = U\tilde{F}$ as in Lemma 3.8. Assume $\alpha_0 = 0$ in (3.9). Then one has power series expansions $U \in 1 + TR[[T]]$ and $\tilde{F} \in 1 + T^{-1}R[[T^{-1}]]$, and $v_{r_0}(\tilde{F} - 1) \geq 0$. Let $m = m_{\mathrm{Swan}}(r_0)$ for f as in Corollary 2.20(ii), and assume that $p \nmid m$. Suppose that $(f^{\mathrm{an}})^{-1}(D[r_0])$ is connected. Furthermore, suppose that we know that the left-slope of δ_F is bounded above by some $s_U \in \mathbb{Z}_{>0}$ such that $-s_U < m$ for all $r \in (0, r_0]$. Let the c_i be as in (3.19) and (3.20), relative to $s = s_U$ for U and $s = m$ for $\tilde{F}$. If $m > 0$, then*

$$\lambda_{m,r_0}(F) = \min\left(r_0,\ \max\left(\mu_{s_U,m}(U, \tilde{F}), \frac{v(c_m) - p/(p-1)}{m}\right)\right).$$

If $m = -1$, then

$$\lambda_{m,r_0}(F) = \min\left(r_0, \mu_{s_U,m}(U, \tilde{F})\right).$$

Proof We pick $r \in (0, r_0)$ and analyze the left-slope of δ_F at r. By Corollary 2.20(iv), we have $m_{\mathrm{Swan}}(r) = |\mathbb{B}[r]| - 1 = |\mathbb{B}[r_0]| - 1$, so $m = m_{\mathrm{Swan}}(r)$ for all $r \in (0, r_0]$. By Proposition 3.14, we have $\delta_F(r) < p/(p-1)$. Note that, since $0 < r < r_0$, we have $v_r(U-1) > 0$ and $v_r(\tilde{F} - 1) > 0$. By Proposition 3.1, both $\delta_{\tilde{F}}(r)$ and $\delta_U(r)$ are less than $p/(p-1)$.

By applying Corollary 2.18 to some r' slightly larger than r, we see that the right-slope of $\delta_{\tilde{F}}$ at r is at most m. Thus, Proposition 2.13 shows that $\mathrm{ord}_{\bar{0}}(\omega_{\tilde{F}}(r)) \geq -m - 1$ (if $\omega_{\tilde{F}}(r)$ exists, that is, if $\delta_{\tilde{F}}(r) > 0$). By Corollary 3.18, we know that $\delta_{\tilde{F}}(r)$ and $\omega_{\tilde{F}}(r)$ (if it exists) can be read off from the c_i for $0 < i \leq m$.

Since the left-slope of δ_F at r is at most s_U, Proposition 2.13 shows that $\mathrm{ord}_{\bar{\infty}}(\omega_F(r)) \geq -s_U - 1$. If $\mathrm{ord}_{\bar{\infty}}(\omega_U(r)) < -s_U - 1$, then Proposition 3.3 shows that $\delta_U(r) > \delta_{\tilde{F}}(r)$ and thus $\omega_F(r) = \omega_{\tilde{F}}(r)$ and $\delta_{\tilde{F}}(r) = \omega_{\tilde{F}}(r)$. By Proposition 3.1, this means that $\omega_F(r)$ can be read off from the c_i for $0 < i \le m$. If $\mathrm{ord}_\infty(\omega_U(r)) \ge -s_U - 1$, then Corollary 3.18 shows that $\delta_U(r)$ and $\omega_U(r)$ (if it exists) can be read off from the c_i for $-s_U \le i < 0$. In all cases, using Proposition 3.3, $\delta_F(r)$ and $\omega_F(r)$ (if it exists) can be read off from the c_i for $-s_U \le i \le m$.

By Remark 3.6, it suffices to show, for all $r \in \mathbb{Q} \cap (0, r_0)$, that the left-slope of δ_F at r is equal to m iff

$$\mu_{s_U,m}(\mathcal{U}, \tilde{F}) < r \quad \text{and} \quad \frac{\nu(c_m) - p/(p-1)}{m} < r.$$

By Corollary 2.18, the left-slope of δ_F is m iff $\mathrm{ord}_\infty(\omega_F(r)) = -m+1$ and $\delta_F(r) > 0$. Using Propositions 3.1 and 3.3, we see that this happens iff the $c_m T^{-m}$ term "dominates" at r among all the terms $c_i T^{-i}$ in the range above and $\nu_r(c_m T^{-m}) < p/(p-1)$. Specifically,

$$\nu(c_m) - mr \le \nu(c_i) - ir$$

for all $i \in \{-s_U, \ldots, m\} \backslash \{0\}$ and $\nu(c_m) - mr < p/(p-1)$. This is equivalent to $\mu_{s_U,m}(\mathcal{U}, \tilde{F}) < r$ and $(\nu(c_m) - p/(p-1))/m < r$ when $m > 0$.

Since $(f^{\mathrm{an}})^{-1}(D[r_0])$ is connected, so is $(f^{\mathrm{an}})^{-1}(D(r)$. When $m = -1$ ($|\mathbb{B}[r]| = 0$), this implies that $\delta_F(r) > 0$ by Lemma 2.17. Thus $\nu_r(c_m T^{-m}) < p/(p-1)$ automatically when the $c_m T^{-m}$ term dominates. This shows that the left-slope of δ_F is m iff $\mu_{s_U,m}(\mathcal{U}, \tilde{F}) < r$. □

4 Relative cyclic covers

Let $\mathcal{A}$ be a rigid-analytic space over K. Throughout this section, if P is any mathematical object over a subset $\mathcal{S}$ of $\mathcal{A}$ and $a \in \mathcal{S}$, we write P_a for its restriction above a. When we say that an object P over $\mathcal{A}$ has a certain property *locally on* $\mathcal{A}$, we mean that there exists a flat, surjective, qcqs morphism $\mathcal{A}' \to \mathcal{A}$ such that the pullback of P to $\mathcal{A}'$ has this property. If $\mathcal{A}$ is qcqs, then it is no restriction to assume that $\mathcal{A}'$ is a finite disjoint union of affinoids.

4.1 Relative open disks

Let $\mathcal{X} \to \mathcal{A}$ be a relative smooth and proper curve.

Definition 4.1 An admissible open subset $\mathcal{D} \subset \mathcal{X}$ is called a *relative open disk* if locally on $\mathcal{A}$ the following holds.

(i) There exists an affinoid subdomain $\mathcal{U} \subset \mathcal{X}$ containing $\mathcal{D}$ such that the morphism $\mathcal{U} \to \mathcal{A}$ extends to a formally smooth morphism $\mathcal{U}_R \to \mathcal{A}_R$ of formal models with special fiber $\bar{U} \to \bar{A}$.

(ii) There exists a section $\sigma : \mathcal{A}_R \to \mathcal{U}_R$ such that

$$\mathcal{D} =]\bar{\Sigma}[_{\mathcal{U}_R}$$

is the formal fiber of the closed subset $\bar{\Sigma} := \sigma(\bar{A}) \subset \bar{U}$.

Lemma 4.2 *Let $\mathcal{D} \subset \mathcal{X}$ be a relative open disk. Then locally on $\mathcal{A}$ there exists an affinoid neighborhood $\mathcal{U} \subset \mathcal{X}$ of $\mathcal{D}$ and a regular function $T \in \mathcal{O}(\mathcal{U})$ such that*

(i) *$\mathcal{D} \subset \mathcal{X}$ is defined by the condition $|T| < 1$, and*

(ii) *for all $a \in \mathcal{A}$ the fiber $\mathcal{D}_a \subset \mathcal{X}_a$ is an open disk with parameter T.*

Proof We may assume that $\mathcal{A}$ is an affinoid domain and that there exists an affinoid subdomain $\mathcal{U} \subset \mathcal{X}$ as in Definition 4.1. By assumption, $\mathcal{U} \to \mathcal{A}$ extends to a formally smooth morphism between formal models $\mathcal{U}_R \to \mathcal{A}_R$ with a section $\sigma: \mathcal{A}_R \to \mathcal{U}_R$ such that $\mathcal{D}$ is the formal fiber of the image of $\bar{A}$ under σ. It follows from [17], Lemma 1.2.2 that $\Sigma := \sigma(\mathcal{A}_R) \subset \mathcal{U}_R$ is an effective relative Cartier divisor of degree one. This means that locally on $\mathcal{A}_R$, and after shrinking $\mathcal{U}_R$, there exists $T \in \mathcal{O}(\mathcal{U}_R)$ such that $\Sigma = (T)$ is the principal divisor defined by T. It is clear that T has exactly the properties stated in the lemma. □

4.2 Relative *G*-covers

Let q be a prime number, which may or may not be equal to p. In this section, we will analyze certain families of $\mathbb{Z}/q$-covers of curves, parameterized by $\mathcal{A}$.

Definition 4.3 Let $\mathcal{X} \to \mathcal{A}$ be a relative smooth proper curve and G a finite group. A *relative G-cover* of $\mathcal{X} \to \mathcal{A}$ is a morphism $\mathcal{F}: \mathcal{Y} \to \mathcal{X}$ of rigid-analytic K-spaces with the following properties.

(i) The morphism $\mathcal{F}$ is finite and flat of degree $|G|$.
(ii) The group G acts on $\mathcal{Y}$ in such a way that $\mathcal{X} = \mathcal{Y}/G$,
(iii) There exists a horizontal divisor $\mathcal{S} \subset \mathcal{X}$ such that $\mathcal{S} \to \mathcal{A}$ is finite and étale and $\mathcal{F}$ is étale over $\mathcal{X} \backslash \mathcal{S}$.

Proposition 4.4 *Let $\mathcal{F}: \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/q$-cover.*

(i) *$\mathcal{Y} \to \mathcal{A}$ is a relative smooth and proper curve.*
(ii) *Locally on $\mathcal{A}$ there exist a horizontal divisor $\mathcal{S} \subset \mathcal{X}$ and a regular function $\Phi \in \mathcal{O}(\mathcal{U})$ on $\mathcal{U} := \mathcal{X} \backslash \mathcal{S}$ such that $\mathcal{F}^{-1}(\mathcal{U}) \to \mathcal{U}$ can be identified with the Kummer cover given by the equation*

$$y^q = \Phi.$$

We call Φ a Kummer representative for $\mathcal{F}$.
(iii) *We can choose Φ above locally on $\mathcal{A}$ so that Φ_a is a Kummer representative for $\mathcal{F}_a$ in the sense of Definition 3.4 for all $a \in \mathcal{A}$.*

Proof To prove (i) we note that $\mathcal{Y} \to \mathcal{A}$ is flat because $\mathcal{Y} \to \mathcal{X}$ and $\mathcal{X} \to \mathcal{A}$ are. It therefore suffices to show that every fiber $\mathcal{Y}_a$, $a \in \mathcal{A}$, is a smooth and proper curve. This follows from the classical theory of tame ramification for algebraic curves (the point of this argument is that the notion of flatness in the context of rigid-analytic spaces has all the usual properties, like being stable under base change. This is quite non-trivial, and is proved in [5]).

For the proof of (ii), we look at the coherent $\mathcal{O}_{\mathcal{X}}$-algebra $\mathcal{F}_* \mathcal{O}_{\mathcal{Y}}$. By assumption (i) in Definition 4.3, it is a locally free $\mathcal{O}_{\mathcal{X}}$-module, and then Assumption (ii) shows that we have a G-eigenspace decomposition

$$\mathcal{F}_* \mathcal{O}_{\mathcal{Y}} = \bigoplus_{i=0}^{q-1} \mathcal{L}_i$$

into line bundles $\mathcal{L}_i$, with $\mathcal{L}_0 = \mathcal{O}_{\mathcal{X}}$. Here $\mathcal{L}_i$ is the eigenspace for the character $G \to K^\times$, $n \mapsto \zeta_q^{in}$, for some fixed qth root of unity $\zeta_q \in K$. Multiplication induces embeddings

$$\mathcal{L}_i \otimes \mathcal{L}_j \hookrightarrow \mathcal{L}_{i+j \pmod q}. \tag{4.5}$$

In particular, we obtain an embedding

$$\mathcal{L}_1^{\otimes q} \hookrightarrow \mathcal{O}_{\mathcal{X}}. \tag{4.6}$$

Now let $\mathcal{S} \subset \mathcal{X}$ be a horizontal divisor such that $\mathcal{F}$ is étale over $\mathcal{U} := \mathcal{X} \backslash \mathcal{S}$. Then Kummer theory shows that the embeddings in (4.5) and (4.6) are in fact isomorphisms on $\mathcal{U}$. After restricting the base $\mathcal{A}$ to a suitable affinoid subdomain and after enlarging the horizontal divisor $\mathcal{S}$, we may assume that $\mathcal{L}_1|_{\mathcal{U}}$ is trivialized by a section $y \in \mathcal{L}_1(\mathcal{U})$. Then y^i trivializes $\mathcal{L}_i|_{\mathcal{U}}$ for all i. Furthermore, if Φ is the image of y^q under the isomorphism (4.6), we obtain an identification

$$\mathcal{F}_*\mathcal{O}_{\mathcal{F}^{-1}(\mathcal{U})} = \mathcal{O}_{\mathcal{U}}[y \mid y^q = \Phi]. \tag{4.7}$$

This proves (ii).

For any $a \in \mathcal{A}$, if y_a is the restriction of y over $\mathcal{U}_a := \mathcal{U} \cap \mathcal{X}_a$, then (4.7) shows that $\mathcal{F}_*\mathcal{O}_{\mathcal{F}^{-1}(\mathcal{U}_a)} = \mathcal{O}_{\mathcal{U}_a}[y_a \mid y_a^q = \Phi_a]$. Viewed birationally, this means that Φ_a is a Kummer representative for $\mathcal{F}_a$. This proves (iii). □

Remark 4.8 Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/q$-cover, and let $\mathcal{D} \subset \mathcal{X}$ be a relative open disk. Then (locally on $\mathcal{A}$) we can choose a Kummer representative Φ for $\mathcal{F}$ as in Proposition 4.4 whose restriction to $\mathcal{D}$ is regular and power bounded, i.e., belongs to the ring

$$\mathcal{O}^\circ(\mathcal{D}) = \left\{ f \in \mathcal{O}(\mathcal{D}) \mid |f(x)| \le 1 \ \forall x \in \mathcal{D} \right\}.$$

In particular, if $\mathcal{A} = \mathrm{Sp}\,A$ is an affinoid and T is a parameter for $\mathcal{D}$, then $\Phi \in A^\circ[[T]]$.

Since the remark will not be needed in the sequel, we only sketch the proof, which proceeds by looking again at the proof of Proposition 4.4. The Kummer representative Φ comes from a trivialization of the line bundle $\mathcal{L}_1$. It is easy to see that, locally on $\mathcal{A}$, every line bundle on $\mathcal{X}$ can be trivialized on an affinoid neighborhood of $\mathcal{D}$. If we use this trivialization to define Φ, then Φ is automatically regular on an affinoid neighborhood of $\mathcal{D}$. In particular, Φ is bounded on $\mathcal{D}$. After multiplying Φ with a suitable constant, we may then assume that Φ is power bounded.

Proposition 4.9 *Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/q$-cover over $\mathcal{A}$ and let $\mathcal{D} \subset \mathcal{X}$ be a relative open disk. If the inverse image $C_a := \mathcal{F}_a^{-1}(D_a)$ is an open disk for all $a \in \mathcal{A}$, then $\mathcal{C} := \mathcal{F}^{-1}(\mathcal{D}) \subset \mathcal{Y}$ is a relative open disk.*

Proof We may assume that $\mathcal{A}$ is an affinoid and that there exist an affinoid neighborhood $\mathcal{U} \subset \mathcal{X}$ of $\mathcal{D}$ and a parameter T for $\mathcal{D}$ as in Lemma 4.2. Let $\Sigma \subset \mathcal{D}$ denote the relative divisor given by $T = 0$. Then Σ is the image of a section $\mathcal{A} \to \mathcal{D}$, by construction, and $\mathcal{A}' := \mathcal{F}^{-1}(\Sigma) \to \mathcal{A}$ is a finite flat covering of degree q. Replacing $\mathcal{A}$ with $\mathcal{A}'$, we may assume that there exists a section $\mathcal{A} \to \mathcal{F}^{-1}(\Sigma)$. Let $\Sigma' \subset \mathcal{C}$ denote its image.

Since $\mathcal{F}$ is finite, $\mathcal{V} := \mathcal{F}^{-1}(\mathcal{U})$ is an affinoid subdomain of $\mathcal{Y}$. The finite morphism $\mathcal{V} \to \mathcal{U}$ extends to a finite morphism between the canonical formal R-models, $\mathcal{V}_R \to \mathcal{U}_R$. Let $\mathcal{A}_R$ be the canonical formal model of $\mathcal{A}$. By the reduced fiber theorem [6, p. 362], there exist a rig-étale covering $\mathcal{A}'_R \to \mathcal{A}_R$ and a finite rig-isomorphism

$$\mathcal{V}'_R \to \mathcal{V}_R \times_{\mathcal{A}_R} \mathcal{A}'_R$$

such that $\mathcal{V}'_R \to \mathcal{A}'_R$ is flat and has geometrically reduced fibers. Since rig-étale morphisms in the context of [6] are qcqs, we may, for the proof of the proposition, assume that there exists a finite R-model $\mathcal{V}_R$ of $\mathcal{V}$ such that $\mathcal{V}_R \to \mathcal{A}_R$ is flat and has geometrically reduced fibers. Let $\bar{V}$ be the special fiber of $\mathcal{V}_R$, and let $\bar{\Sigma}' \subset \bar{V}$ denote the intersection of $\bar{V}$ with the closure of $\Sigma' \subset \mathcal{V}$ in $\mathcal{V}_R$. Then

$$\mathcal{C} =]\bar{\Sigma}'[_{\mathcal{V}_R}.$$

We have to prove that $\mathcal{V}_R \to \mathcal{A}_R$ is formally smooth along $\bar{\Sigma}'$. Because $\mathcal{V}_R \to \mathcal{A}_R$ is flat, it suffices to prove that all fibers of $\bar{V} \to \bar{A}$ over all closed points of $\bar{A}$ are smooth in a neighborhood of $\bar{\Sigma}'$.

Let $\bar{a} \in \bar{A}$ be a closed point and $a_R\colon \mathrm{Spf}\, R' \to \mathcal{A}_{R'}$ a lift of $\bar{a}$, where R' is a discrete valuation ring which is a finite extension of R (such a lift exists by [3, §8.3, Proposition 8]). Let $V_{a,R'} \to \mathrm{Spf}\, R'$ denote the fiber of $\mathcal{V}_R \to \mathcal{A}_R$ over $a_{R'}$. By construction, $V_{a,R'}$ is an admissible formal R'-scheme whose generic fiber is smooth affinoid curve and whose special fiber $\bar{V}_{\bar{a}}$ is equal to the fiber of $\bar{V} \to \bar{A}$ over $\bar{a}$. Let $\bar{\Sigma}'_{\bar{a}} \in \bar{V}_{\bar{a}}$ denote the intersection of $\bar{\Sigma}'$ with $\bar{V}_{\bar{a}}$ (a closed point). The main assumption of the proposition says that the formal fiber $D_a =]\bar{\Sigma}'_{\bar{a}}[_{V_{a,R'}}$ is an open disk. It follows from [1], Proposition 3.4, that $\bar{V}_{\bar{a}}$ is smooth in a neighborhood of $\bar{\Sigma}'_{\bar{a}}$ (it is here that we use that the special fiber $\bar{V}_{\bar{a}}$ is reduced). This completes the proof of the proposition. □

4.3 Assumptions on relative covers

The key assumption on relative G-Galois covers we need is the following:

Assumption 4.10 There is a subset $\mathcal{D} \subset \mathcal{X}$ that is a relative open disk above $\mathcal{A}$. For each affinoid $\mathcal{B}$ with a surjective qcqs map to an affinoid in an admissible cover of $\mathcal{A}$ such that the pullback of $\mathcal{D}$ to $\mathcal{B}$ is a trivial family of disks, we pick a function T on the pullback of $\mathcal{X}$ to $\mathcal{B}$ as in Lemma 4.2. This is a simultaneous parameter (Sect. 2.5.1) on all the fibers $\mathcal{D}_a$ for $a \in \mathcal{B}$. We identify all $\mathcal{D}_a$ with (the same) open disk D and use the notation of Sect. 2.5 where appropriate. We identify the pullback of $\mathcal{D}$ to $\mathcal{B}$ with $\mathcal{B} \times D$.

Remark 4.11 Note that Assumption 4.10 holds trivially if $\mathcal{X} \to \mathcal{A}$ is a trivial family, as it is in the introduction. However, it is important to prove our results under the generality of Assumption 4.10 in order to facilitate an induction from $\mathbb{Z}/p$-covers to more general ones.

We make some further assumptions and notation for the remainder of Sect. 4.

Assumption 4.12 (i) For each $\mathcal{B}$ as in Assumption 4.10, there exists $s_1 \in \mathbb{Q}_{>0}$ such that for all $a \in \mathcal{B}$ and $r \in (0, s_1)$, the set $\mathcal{F}_a^{-1}(D(r))$ is connected.

(ii) If $\mathcal{S}$ is as in Definition 4.3 and $\mathcal{D}$ is as in Assumption 4.10, then $\mathcal{S} \cap \mathcal{D} \to \mathcal{A}$ is finite étale of some degree N (in particular, after pulling back to some $\mathcal{B}$ as in Assumption 4.10, the number of branch points of $\mathcal{F}_a$ in D over $\bar{K}$ is N for all $a \in \mathcal{B}$).

Proposition 4.13 *Under Assumptions 4.10 and 4.12, with $\mathcal{B}$ as in Assumption 4.10, there exists $s_2 \in \mathbb{Q}_{>0}$ such that for each $a \in \mathcal{B}$, the cover $\mathcal{F}_a$ has no branch point in $D \backslash D[s_2]$.*

Proof Let $\mathcal{S}$ be as in Definition 4.3, and let $\mathcal{S}_{\mathcal{B}}$ be its pullback to $\mathcal{B}$. The projection $\pi : \mathcal{S} \cap (\mathcal{B} \times D) \to D$ is an analytic function. Since $\mathcal{S}$ is finite over $\mathcal{B}$, it is affinoid. By the maximum principle, as s ranges through $\mathcal{S}_{\mathcal{B}}$, the function $v(\pi(s))$ achieves its minimum. This is the s_2 we seek. □

Remark 4.14 (i) If $\mathcal{A}$ is qcqs, then one can choose finitely many $\mathcal{B}$ as in Assumption 4.10 that completely cover $\mathcal{A}$. In particular, one can choose a *uniform* s_1 and s_2 above that work for all $\mathcal{B}$.

(ii) Under Assumptions 4.10 and 4.12, if $G = \mathbb{Z}/p$ with χ a degree 1 character on $\mathbb{Z}/p$, and $\mathcal{B}$ is as in Assumption 4.10, we can define functions δ_χ and ω_χ for each $\mathcal{F}_a$, $a \in \mathcal{B}$, as in Sect. 2.5.3. These functions descend to $\mathcal{A}$, so by abuse of notation, we can consider $a \in \mathcal{A}$ instead of $\mathcal{B}$.

(iii) If $G \cong \mathbb{Z}/p$, then Lemma 2.16 shows that Assumption 4.12(i) holds automatically whenever S is non-empty (just take any $s_1 < s_2$).

Notation 4.15 If $\mathcal{A}$ is qcqs, we will generally define $r_0 = \min(s_1, s_2)$, with s_1 chosen uniformly as in Assumption 4.10 and s_2 chosen uniformly as in Proposition 4.13. In particular, $\mathcal{F}_a^{-1}(D(r))$ is connected for all $a \in A$ and all $r \in (0, r_0) \cap \mathbb{Q}$.

Remark 4.16 If $\mathcal{S}$ is as in Definition 4.3 and d is the degree of $\mathcal{S} \to \mathcal{A}$, then the number of branch points in $\mathcal{F}_a$ lying outside $\mathcal{D}$ is bounded above by d.

4.4 Variation of Kummer representatives: main results in the $\mathbb{Z}/p$ case

Let $\mathcal{A}$ be a rigid-analytic space, and let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/p$-cover of $\mathcal{X} \to \mathcal{A}$. We work under Assumptions 4.10 and 4.12. Let $\pi\colon \mathcal{D} \to \mathcal{A}$ be the relative open disk from Assumption 4.10. Let $\mathcal{B}$ be an affinoid as in Assumption 4.10, and shrink $\mathcal{B}$ to a smaller affinoid on which there exists a Kummer representative Φ for the pullback of the restriction of $\mathcal{F}$ above $\mathcal{D}$ as in Proposition 4.4(ii).

Lemma 4.17 *In the context above, after a possible finite extension of K, there exist meromorphic functions U and $\tilde{F}$ on $\mathcal{D} \times_{\mathcal{A}} \mathcal{B}$ such that U is a unit on $\mathcal{D} \times_{\mathcal{A}} \mathcal{B}$, that $\tilde{F}_a$ is of the form* (3.9) *for all $a \in \mathcal{B}$, and that $F_a := U_a\tilde{F}_a$ and Φ_a differ only by multiplication by a pth power of a rational function on $D = \mathcal{D}_a$. In particular, F_a is a Kummer representative for $\mathcal{F}_a$ when restricted to D.*

Proof For each $a \in \mathcal{B}$, Assumptions 4.10 and 4.12 show that Φ_a has N zeroes with prime-to-p order lying in $D[r_0]$, as well as some number of zeroes with order divisible by p lying in D. Let $\tilde{F}_a$ be a polynomial in T whose zeroes are the same as the zeroes of order not divisible by p of Φ_a, with the same multiplicities $\pmod p$, such that all the multiplicities are between 0 and $p-1$. Let Q_a be a polynomial in T such that $\tilde{F}_aQ_a$ has the same zeroes and multiplicities as Φ_a. Then all multiplicities of zeroes of Q_a are divisible by p. Let $U_a = \Phi_a/\tilde{F}_aQ_a$. After possibly multiplying $\tilde{F}_a$ by a constant and a power of T^p and adjusting Q_a accordingly so that U_a stays fixed, we get that $\tilde{F}_a$ is in the form of (3.9) and that Q_a is a pth power (this may require an extension of K).

Since Φ is analytic, its zeroes and poles vary analytically in $\mathcal{B}$. Thus $\tilde{F}_a$ extends to an analytic function $\tilde{F}$ on $\mathcal{D} \times_{\mathcal{A}} \mathcal{B}$. Since Assumption 4.12(ii) shows that the poles and zeroes of Φ_a never collide, they have "constant" orders as they vary over $\mathcal{B}$. Thus, Q_a (and U_a) also extend to analytic functions Q and U on $\mathcal{D} \times_{\mathcal{A}} \mathcal{B}$. Then U is a unit because U_a is for all $a \in \mathcal{B}$.

The last assertion follows from Proposition 4.4(iii). □

Remark 4.18 If U and $\tilde{F}$ are as in Lemma 4.17, then for each $a \in \mathcal{B}$, the functions U_a and $\tilde{F}_a$ are of the forms of Remarks 3.11 and 3.12, respectively.

The following lemma contains the key result from rigid geometry that makes everything work. We need a bit of setup. Let $\mathcal{B} = \operatorname{Sp} B$ be an affinoid, and let $U, \tilde{F} \in B[[T]]$ be such that, for each $a \in \mathcal{B}$, the functions U_a and $\tilde{F}_a$ are in the form of Remarks 3.11 and 3.12, respectively, with $\alpha_0 = 0$ in Remark 3.12. For any $s_U, m \in \mathbb{Z}_{>0}$ with $p \nmid m$ and $-s_U < m$, let the $c_{i,a}$ for $-s_U \leq i \leq m$ be computed from U_a and $\tilde{F}_a$ as in (3.19) and (3.20). By Lemma 3.16, there is a finite, flat cover $\pi : \mathcal{C} \to \mathcal{B}$ such that the $c_{i,a}$ give analytic functions c_i on $\mathcal{C}$, and $\nu(c_i)$ factors through this cover to give a well-defined function on $\mathcal{B}$. Recall that $\mu_{s_U,m}(U_a, \tilde{F}_a)$ was defined in Definition 3.21.

Lemma 4.19 (cf. [19, Lemma 6.16]) *In the situation above, the function*

$$\max\left(\mu_{s_U,m}(U_a, \tilde{F}_a), \frac{\nu(c_{m,a}) - p/(p-1)}{m}\right)$$

achieves its minimum as a ranges over $\mathcal{B}$. Furthermore, the subset of $\mathcal{B}$ on which the minimum is attained is qcqs. The same holds for the function $\mu_{s_U,m}(U_a, \tilde{F}_a)$.

Proof Let S be the set of integers i satisfying $s_U \leq i < m$ and $p \nmid i$. Let $\pi' : \mathcal{C}' \to \mathcal{C}$ be a finite, flat cover on which the functions $g_i := \sqrt[m-i]{c_i/c_m}$ for $i \in S$ and $g_m := \sqrt[m]{p^{p/(p-1)}/c_m}$ are defined as meromorphic functions (take a finite extension of K if necessary). Since $\nu(g_i)$ descends to a function on $\mathcal{B}$ for all i and images of qcqs rigid-analytic spaces under flat morphisms to qcqs spaces are qcqs [5, Corollary 5.11], it suffices to show that if

$$\gamma = \sup_{a \in \mathcal{C}'}\left(\min_{i \in S \cup \{m\}}(\nu(g_{i,a}))\right) \geq 0,$$

then γ is achieved on a qcqs subset of $\mathcal{C}'$.

In particular, we may assume that $\gamma \neq -\infty$. Pick $a \in \mathcal{C}'$ such that $\gamma' := \min_{i \in S \cup \{m\}}(\nu(g_{i,a})) \neq -\infty$. We may then replace $\mathcal{C}'$ with the qcqs Weierstrass domain given by $\nu(g_m) \geq \gamma$. In particular, we may assume that c_m^{-1}, and thus all the g_i, are *analytic* on $\mathcal{C}'$.

The proof now parallels that of [2, §7.3.4, Lemma 7], translated into valuation-theoretic language. Observe that $g_m \neq 0$ on $\mathcal{C}'$, so the g_i have no common zero. For each $j \in S \cup \{m\}$, let $\mathcal{C}'_j \subseteq \mathcal{C}$ be the rational subdomain where $\nu(g_j)$ is minimal among all the g_i for $i \in S \cup \{m\}$. Then, the restriction of $\min_{i \in S \cup \{m\}}(\nu(g_i))$ to $\mathcal{C}'_j$ is simply equal to $\nu(g_j)$, which attains its minimum on $\mathcal{C}'_j$ by the maximum modulus principle. Furthermore, the subspace of $\mathcal{C}'_j$ where this minimum is attained is a Weierstrass domain in $\mathcal{C}'_j$, which means it is qcqs. Thus, the subspace of $\mathcal{C}'$ where γ is attained is a union of finitely many qcqs spaces. Since $\mathcal{C}'$ is affinoid, being a finite cover of an affinoid, this union is qcqs, completing the proof of the first statement.

The last statement follows by replacing $S \cup \{m\}$ with S everywhere. □

Let $\mathcal{A}$ be a qcqs rigid-analytic space over K and let $\mathcal{F} : \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/p$-cover of $\mathcal{X} \to \mathcal{A}$ satisfying Assumptions 4.10 and 4.12, with N as in Assumption 4.12(ii) and r_0 as in Notation 4.15. By abuse of notation, we define λ_{N-1,r_0} as a function from $\mathcal{A}$ to $[0, r_0]$ (specifically, if $a \in \mathcal{A}$, then $\lambda_{N-1,r_0}(a)$ is $\lambda_{N-1,r_0}(\mathcal{F}_a)$ as defined in Definition 3.5).

The following is the main result of this section.

Proposition 4.20 *In the situation above, Let $\gamma = \inf_{a \in \mathcal{A}}(\lambda_{N-1,r_0}(a))$. Then the subset of $\mathcal{A}$ where $\lambda_{N-1,r_0}(a) = \gamma$ is a non-empty qcqs set.*

Proof Since $\mathcal{A}$ is qcqs and there exist Kummer representatives locally on $\mathcal{A}$ (Proposition 4.4, Remark 4.8), we can reduce to the case that $\mathcal{A}$ is affinoid with Kummer representative Φ of $\mathcal{F}$. Furthermore, since images of qcqs spaces under flat morphisms with quasi-separated codomain are qcqs [5, Corollary 5.11], and finite unions of qcqs spaces inside an ambient quasi-separated space are also qcqs, we may assume that the relative open disk $\mathcal{D}$ from Assumption 4.10 is already trivial over $\mathcal{A}$. As in Assumption 4.10, we identify each $\mathcal{D}_a$ with the open disk D.

For each $a \in \mathcal{A}$, the cover $\mathcal{F}_a$ has N branch points in $D[r]$ for all $r \in (0, r_0]$. By Corollary 2.20(iv), we have $m_{\mathrm{Swan}}(r) = N - 1$ for each $\mathcal{F}_a$ and all such r.

Let U, $\tilde{F}$ be as in Lemma 4.17. Then $F_a := U_a\tilde{F}_a$ is a Kummer representative for $\mathcal{F}_a$ restricted to D. The functions $\tilde{F}_a$ are all in the form of Remark 3.12 for *fixed* values of the α_i. We consider the cases $\alpha_0 = 0$ and $\alpha_0 \neq 0$ separately.

If $\alpha_0 \neq 0$, then Proposition 3.14 shows that $\delta_{F_a}(r) = p/(p-1)$ for all $a \in \mathcal{A}$ and all $r \in (0, r_0]$. By definition, $\lambda_{N-1,r_0}(a)$ equals 0 if $N = 1$ or r_0 if $N \neq 1$, independent of a. In both cases, the subset of $\mathcal{A}$ where $\lambda_{N-1,r_0} = \gamma$ is $\mathcal{A}$ itself, which finishes the proof.

Now suppose $\alpha_0 = 0$. If $N \equiv 1 \pmod p$, then δ_{F_a} can only have a left-slope at r equal to $N-1$ if $\delta_{F_a}(r) = p/(p-1)$ or 0, in which case that slope is zero (Proposition 3.7). But Proposition 3.14 shows that $\delta_{F_a}(r) < p/(p-1)$ for all $a \in \mathcal{A}$ and all $r \in (0, r_0)$, so assume $\delta_{F_a}(r) = 0$ on $(0, r_0)$. Now, $N \neq 1$ by Corollary 3.15. If $N > 1$, then we have $\lambda_{N-1,r_0}(a) = r_0$ independent of a, finishing the proof. So we may assume $p \nmid (N-1)$.

By Proposition 2.14, we have that $\mathrm{ord}_\infty(\omega_{F_a}(r)) \geq -2pg_X/(p-1) - d$, where g_X is the genus of any $\mathcal{X}_a$ and d is as in Remark 4.16. Pick $s_U \in \mathbb{Z}_{>0}$ such that $s_U > 2pg_X/(p-1) + d - 1$. Proposition 2.13 implies that the left-slope of δ_{F_a} at r is bounded above by s_U. We can now apply Proposition 3.22 to see that, if $N > 0$, then

$$\lambda_{N-1,r_0}(\mathcal{F}_a) = \max\left(\mu_{s_U,N-1}(U_a, \tilde{F}_a), (\nu(c_{m,a}) - p/(p-1))/m\right)$$

for all $a \in \mathcal{A}$ for which the right-hand side is less than or equal to r_0. By Lemma 4.19, the right-hand side attains its minimum on a non-empty qcqs subdomain. This minimum must be $\gamma \leq r_0$. So $\lambda_{N-1,r_0}(\mathcal{F}_a)$ also attains its minimum on a non-empty qcqs subdomain. If $N = 0$, then one repeats the same argument with $\mu_{s_U,N-1}(U_a, \tilde{F}_a)$ in place of $\max(\mu_{s_U,N-1}(U_a, \tilde{F}_a), (\nu(c_{m,a}) - p/(p-1))/m)$. □

Corollary 4.21 *Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/p$-cover of $\mathcal{X} \to \mathcal{A}$ where $\mathcal{A}$ is a qcqs rigid-analytic space. Suppose $\mathcal{F}$ satisfies Assumptions 4.10 and 4.12, and let N be as in Assumption 4.12(ii) and r_0 be as in Notation 4.15. Let $r_1, r_2, \ldots$ be a sequence decreasing to 0 such that for each i, there exists $a_i \in A$ such that $\mathcal{F}_{a_i}^{-1}(D[r_i])$ is a closed disk. Then there is a non-empty qcqs subdomain $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{F}_a^{-1}(D)$ is an open disk for every $a \in \mathcal{B}$.*

Proof By Corollary 2.20(iv), we have that $m_{\mathrm{Swan}}(r) = N - 1$ for all $\mathcal{F}_a$ and all $r \in (0, r_0]$.

It follows from Corollary 2.18 that $\lambda_{N-1,r_0}(a_i) < r_i$ for each i. Proposition 4.20 now shows that there exists a qcqs subdomain $\mathcal{B} \subseteq \mathcal{A}$ such that $\lambda_{N-1,r_0}(a) = 0$ for all $a \in \mathcal{B}$. By Corollary 2.18, $\mathcal{F}_a^{-1}(D[r])$ is a disk for all $r \in (0, r_0)$ and $a \in \mathcal{B}$. The corollary then follows by Lemma 2.15. □

4.5 Main results in the $\mathbb{Z}/\ell$ case

Throughout this section, ℓ is a prime number not divisible by p. It is much simpler to understand $\mathbb{Z}/\ell$-covers than $\mathbb{Z}/p$-covers.

Proposition 4.22 *Let D be an open (resp. closed) disk over K, and let $f\colon E \to D$ be a $\mathbb{Z}/\ell$-cover of D. Then E is an open (resp. closed) disk iff f has exactly one branch (equiv. ramification) point.*

Proof Let T be a coordinate making D a unit disk. If f has exactly one branch point, then we can assume it is $T = 0$. Since $D\backslash\{0\}$ has prime-to-p fundamental group $\hat{\mathbb{Z}}/\mathbb{Z}_p$, we may assume f is given by extracting an mth root of T, which clearly yields an appropriate disk.

For the "only if" direction, let σ be an automorphism of E with order ℓ. If E is an open disk, then by [12, Corollary 2.4 and §2.5], after a change in coordinates, σ is given by multiplying by an ℓth root of unity. Thus σ has one fixed point. If E is a closed disk, then σ acts on the reduction $\mathbb{A}^1_k$ of E and thus must have a unique fixed point $x \in \mathbb{A}^1_k$. In particular, σ acts on the open disk $E^\circ \subset E$ of points reducing to x. As we have seen, this action is multiplication by an ℓth root of unity, up to a change in variables. Since $\sigma|_{E^\circ}$ has one fixed point, the same is true for σ, proving the proposition. $\square$

Now, let $\mathcal{A}$ be a qcqs rigid-analytic space over K, let $\mathcal{X} \to \mathcal{A}$ be a relative smooth projective curve, and let $\mathcal{F} : \mathcal{Y} \to \mathcal{X}$ be a relative $\mathbb{Z}/\ell$-cover. Assume $\mathcal{F}$ satisfies Assumptions 4.10 and 4.12 for some $\mathcal{D}$, D, N as in those assumptions. Let r_0 be as in Notation 4.15. For $a \in \mathcal{A}$, let $\mathcal{F}_a\colon \mathcal{Y}_a \to \mathcal{X}_a$ be the fiber of $\mathcal{F}$ above a.

Corollary 4.23 *For $a \in \mathcal{A}$ and $r \in (0, r_0)$, whether $\mathcal{F}_a^{-1}(D[r])$ is a disk or not does not depend on a or r.*

Proof By Proposition 4.22, $\mathcal{F}_a^{-1}(D[r])$ is a disk iff $N = 1$. $\square$

Corollary 4.24 *Suppose $r_1, r_2, \ldots$ is a sequence decreasing to 0 such that for each i, there exists $a_i \in \mathcal{A}$ such that $\mathcal{F}_{a_i}^{-1}(D[r])$ is a closed disk. Then $\mathcal{F}^{-1}(\mathcal{D})$ is a relative open disk.*

Proof It is immediate from Corollary 4.23 and Lemma 2.15 that $\mathcal{F}_a^{-1}(\mathcal{D})$ is an open disk for every $a \in \mathcal{A}$. We conclude using Proposition 4.9. $\square$

5 The main result

Let $\mathcal{A}$ be a rigid-analytic space over K and let G be a finite group. Let $\mathcal{X} \to \mathcal{A}$ be a relative smooth and proper curve. A *tower of relative Galois covers* of $\mathcal{X} \to \mathcal{A}$ is a finite, flat morphism $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ that is a composition of finitely many relative Galois covers $\mathcal{Y} = \mathcal{Y}_n \to \mathcal{Y}_{n-1} \to \cdots \to \mathcal{Y}_0 = \mathcal{X}$. Assumptions 4.10 and 4.12(i) carry over to relative (towers of) Galois covers without change. The analog of Assumption 4.12(ii) is the statement that for each $1 \le i \le n$, the branch divisor $\mathcal{S}_i$ of $\mathcal{F}_i\colon \mathcal{Y}_i \to \mathcal{Y}_{i-1}$ is finite étale of some degree N_i over $\mathcal{A}$. A (tower of) relative G-cover(s) satisfying these assumptions is called *good*.

A tower of relative Galois covers is called *solvable* if it is composed of Galois covers with solvable Galois groups. A solvable tower of relative Galois covers $\mathcal{F}$ satisfying Assumptions 4.10 and 4.12 has a composition series consisting of relative $\mathbb{Z}/p$- and $\mathbb{Z}/\ell$-covers, where ℓ ranges over primes other than p.

Lemma 5.1 *Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a good tower of relative Galois covers such that there exists $a \in \mathcal{A}$ for which $\mathcal{X}_a$ contains a closed disk E_0 whose inverse image E_1 under $\mathcal{F}_a$ is a closed disk. Then each Galois group in the tower is an extension of a cyclic prime-to-p group by a p-group. In particular, $\mathcal{F}$ is solvable.*

Proof Since the image of a closed disk under a finite, flat morphism is a closed disk, we may assume that $\mathcal{F}$ is a relative G-Galois cover and that G acts faithfully on E_1. Abhyankar's lemma allows us to assume, after a finite extension of K, that G acts with p-power inertia at a uniformizer of K. That is, if $\bar{E}_1 \cong \mathbb{A}^1_k$ is the canonical reduction of E_1, then the subgroup H of G acting trivially on $\bar{E}_1$ is a (normal) p-group. Then, G/H acts faithfully on $\mathbb{A}^1_k$, which means it is a finite group contained in $\mathbb{G}_a \rtimes \mathbb{G}_m$. So $G/H \cap \mathbb{G}_a$ is a p-group, and $(G/H)/(G/H \cap \mathbb{G}_a) \subseteq \mathbb{G}_m$ is cyclic of prime-to-p order. We are done. $\square$

Our main result is the following:

Theorem 5.2 *Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a good tower of relative Galois covers parameterized by a qcqs rigid-analytic space $\mathcal{A}$, and let D be as in Assumption 4.10. Suppose there is a decreasing sequence $r_1, r_2, \ldots$ with limit 0 such that for each i, there exists $a \in \mathcal{A}$ with $\mathcal{F}_a^{-1}(D[r_i])$ a closed disk. Then there is a non-empty qcqs $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{F}_a^{-1}(D)$ is an open disk for all $a \in \mathcal{B}$.*

Proof By Lemma 5.1, we may assume that $\mathcal{F}$ is solvable. We proceed by induction on the length of a composition series for $\mathcal{F}$ with prime order Galois groups. If the length is 1, then the theorem is simply Corollary 4.21 in the case $\mathbb{Z}/p$ or Corollary 4.24 in the case $\mathbb{Z}/\ell$.

Suppose the length is greater than 1. If $\mathcal{Y} = \mathcal{Y}_n \to \mathcal{Y}_{n-1} \to \cdots \to \mathcal{Y}_0 = \mathcal{X}$ is such a composition series for $\mathcal{F}$, let $\mathcal{Z} = \mathcal{Y}_{n-1}$, and let $\mathcal{P}\colon \mathcal{Z} \to \mathcal{X}$ be the canonical morphism. By the induction hypothesis, there exists a qcqs $\mathcal{A}' \subseteq \mathcal{A}$ such that for all $a \in \mathcal{A}'$, the space $\mathcal{P}_a^{-1}(D)$ is an open disk. After replacing $\mathcal{A}$ by $\mathcal{A}'$, we may assume that $\mathcal{P}_a^{-1}(D)$ is a disk for all $a \in \mathcal{A}$. In fact, by Proposition 4.9, $\mathcal{E} := \mathcal{P}^{-1}(\mathcal{D})$ is a relative open disk.

Thus the subcover $\mathcal{Q}\colon \mathcal{Y} \to \mathcal{Z}$ is a good relative $\mathbb{Z}/p$- or $\mathbb{Z}/\ell$-cover with respect to the relative open disk $\mathcal{E}$. By the induction hypothesis again, we have that there is a non-empty qcqs $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{Q}_a^{-1}(\mathcal{P}_a^{-1}(D))$ is an open disk for all $a \in \mathcal{B}$. This is the same as $\mathcal{F}_a^{-1}(D)$, so we are done. $\square$

Part (ii) of the following corollary is a useful result for the local lifting problem. In particular, it plays a key role in proving the main results of [18].

Corollary 5.3 *Let $\mathcal{F}\colon \mathcal{Y} \to \mathcal{X}$ be a tower of good relative Galois covers, with all Galois groups p-groups, parameterized by a qcqs rigid-analytic space $\mathcal{A}$, and let r_0 be as in Notation 4.15. Assume $\mathcal{F}$ is residually purely inseparable at all $r \in (0, r_0] \cap \mathbb{Q}$. Let m_{diff} be as in Corollary 2.20(i). Define $\lambda_{\mathrm{diff}}\colon \mathcal{A} \to [0, r_0]$ by taking $\lambda_{\mathrm{diff}}(a)$ to be the maximum of all $r \in (0, r_0]$ such that the left-slope of $\delta^{\mathrm{Berk}}_{\mathcal{Y}_a/\mathcal{X}_a}$ at r is (strictly) less than m_{diff}, or 0 if there is no such r. If $\mathcal{F}$ is G-Galois and χ is a faithful, irreducible character on G, then define λ_{Swan} in the same way, replacing m_{diff} and $\delta^{\mathrm{Berk}}_{\mathcal{Y}_a/\mathcal{X}_a}$ by m_{Swan} (Corollary 2.20(ii)) and $\delta_{\mathcal{Y}_a/\mathcal{X}_a,\chi}$.*

(i) *There is a non-empty qcqs $\mathcal{B} \subseteq \mathcal{A}$ on which λ_{diff} achieves its minimum.*
(ii) *If G and χ are as above, let $H \subseteq G$ be a cyclic subgroup such that χ is induced from a character of H, and let $H' \subseteq H$ be the unique subgroup of order p. Let $\varphi\colon \mathcal{Y}/H' \to \mathcal{X}$ be the quotient morphism of $\mathcal{F}$ and suppose $(\varphi_a^{\mathrm{an}})^{-1}(D[r])$ is a closed disk for all $a \in \mathcal{A}$ and $r \in (0, r_0] \cap \mathbb{Q}$. Then there is a non-empty qcqs $\mathcal{B} \subseteq \mathcal{A}$ on which λ_{Swan} achieves its minimum.*

Proof By Corollary 2.20(i), for $r \in (0, r_0]$, the left-slope of $\delta^{\mathrm{Berk}}_{\mathcal{Y}_a/\mathcal{X}_a}$ at r is m_{diff} exactly when $\mathcal{F}_a^{-1}(D[r])$ is a closed disk. Combining this with Lemma 2.15, for $r \in (0, r_0)$, we have $\lambda_{\mathrm{diff}}(a) \leq r$ iff $\mathcal{F}_a^{-1}(D(r))$ is an open disk. Thus, if $\gamma = \inf_{a \in \mathcal{A}}(\lambda_{\mathrm{diff}}(a))$, then either $\gamma = r_0$, in which case we are done, or $\lambda_{\mathrm{diff}}(a) = \gamma$ is equivalent to $\mathcal{F}_a^{-1}(D(\gamma))$ being an open disk. By replacing D with $D(\gamma)$ and r_0 with $r_0 - \gamma$, we may assume $\gamma = 0$. Then (i) follows from Theorem 5.2.

The proof of (ii) is exactly the same, using Corollary 2.20(ii) and (iii) in place of Corollary 2.20(i). $\square$

Remark 5.4 Corollary 5.3(i) should also hold without the assumption of pure inseparability, but we do not have a proof at this time, because of the difficulty of generalizing Corollary 2.20 to the non-purely inseparable case.

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